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Towards Optimal Hybrid Control Solutions for Gait Patterns of a Quadruped

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ABSTRACT

We consider the problem of finding optimal gaits for a quadruped robot. Paths are sought which minimize the actuation energy required for walking in an attempt to approximate natural motion. The number of possible gaits for a quadruped is quite large when one considers varied orders of leg motion, different liftoff times, and various ground contact combinations for the legs. The problem is treated as a fully nonlinear optimal hybrid path planning problem on a 22-dimensional state space. Modeling aspects, our numerical approach, and experimental results are discussed in this paper.

1 INTRODUCTION

Hybrid control problems referring to systems containing both continuous and discrete dynamics have received much interest over the past few years (3, 8, 12, 13). They are especially challenging as they may contain switching dynamics at arbitrary times caused by a discrete control input or when the state reaches physical boundaries. Discontinuities in the state are often the result. Nonlinear feedback control methods have considerable difficulty dealing even with small example systems. Traditional path planning methods (2), though better suited for high-dimensional systems, are not equipped to handle the discrete structure in the system. Combinatorial approaches (5) often used for purely discrete problems lack the necessary apparatus for searching a complex, continuous state space.

Optimal quadruped walking is a problem which contains all the properties of a complex hybrid dynamical system. The continuous dynamics of such a multibody system are high-dimensional and extremely nonlinear. These lie in combination with nonlinear algebraic inequality and equality constraints resulting from physical contact with the environment. With each change of the supporting legs, the dynamics switch with possible discontinuities in the state due to collisions with the ground. Unlike the biped case, it is not known a priori which leg will next liftoff or make contact with the ground at any given time. This unknown variability can be designated as a discrete state in the system. With just four legs, the quadruped has a remarkably wide variety of possible leg movements which it may perform. Previous work has concentrated primarily on heuristic strategies or simplified models to derive gait patterns.

We model the quadruped as a tree-structured multibody system and use recursive, symbolic algorithms to calculate the dynamics, collision effects, and kinematic constraints. A similar approach was used for determining minimum energy paths of biped walking (7). A particular gait pattern may be formulated as an optimal control problem where the objective is to minimize the required actuation energy. We solve this problem using the direct collocation method based on a parametrization of the continuous state and control variables and on large-scale nonlinear programming. The generality and power of this type of numerical approach allows us to solve *within* a gait classification for the minimum energy, periodic motion of a quadruped with a full, dynamical model moving in the sagittal plane. Leg liftoff and collision times, stride length, and forward velocity are all continuous parameters which may be optimized. This alone is a challenging hybrid problem with switching dynamics and jump conditions which has not previously been solved. Several different optimal gait patterns are compared and discussed here together with experimental results. A numerical approach is described for the solution of the global hybrid problem in which the optimal order of leg movements is also to be determined.

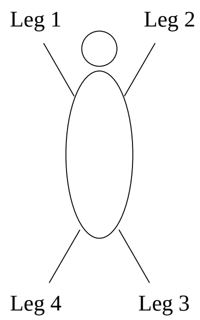
2 HYBRID OPTIMAL CONTROL PROBLEM OF OPTIMAL GAITS

2.1 Quadruped locomotion

Some important definitions which we will make use of are: **stride:** complete cycle of leg movements where each leg has been set down once; **stride length:** distance traveled in a stride; **duty factor:** fraction of duration of a stride for which the leg is in contact with the ground; **relative phase:** stage of the stride at which the leg is set down expressed as the fraction of the duration of the stride following the setting down of an arbitrarily chosen reference leg. We may thus distinguish walks from runs in that walks have duty factors greater than 0.5. Also, symmetric gaits are those for which the left and right legs of a pair have equal duty factors and relative phases differing by 0.5.

Table 1: Common quadruped gaits and relative phases of legs (1).

	Relative Phases of Legs			
Gait Name	Leg 1	Leg 2	Leg 3	Leg 4
amble	0	0.5	0.25	0.75
trot	0	0.5	0	0.5
pace	0	0.5	0.5	0
canter	0	0.3	0	0.7
transverse gallop	0	0.1	0.6	0.5
bound	0	0	0.5	0.5

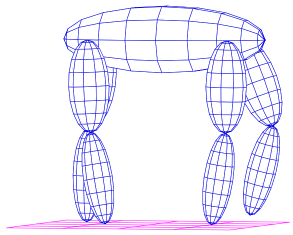


A detailed study of biped and quadruped gaits may be found in the paper by Alexander (1) including a list of some commonly found gaits in animals, see Table 1. The list is not exhaustive, and the relative phases given for the respective gaits are not fixed but may vary depending on the shape and size of the animal. In this paper, we will study only the symmetric gaits amble and trot. Future work will extend our analysis to all possible walks and runs. In nature, symmetric gaits are more common in quadrupeds during walking or slow running. Within the class of symmetric gaits, mammals tend to amble when walking and to trot during slow running, though many exceptions exist. Reptiles, for example, trot even during slow walking (1).

Not given in Table 1 are the duty factors which describe the fraction of time that a leg is in contact with the ground and which may vary considerably depending on the traveling speed.

Table 2: Physical data of the quadruped model.

Link	Mass	Length	Radius
Torso	20 kg	0.5 m	0.12 m
Upper Leg	7 kg	0.32 m	0.07 m
Lower Leg	4 kg	0.32 m	0.05 m



Alone for the gaits we will consider here, at any moment there may be two, three, or four legs in contact with the ground. We denote the separation of a leg with the ground as *liftoff* a leg making ground contact as a *collision*. With each liftoff or collision, a new phase with its respective dynamics begins, and there may occur multiple liftoffs and collisions at a phase transition. After a collision, a leg may enter either a contact state or slipping state, though we will focus only on the contact possibility here.

2.2 Hybrid dynamic model

Our model for the quadruped consists of a 9-link tree-structured multibody system with a central torso and 4 two-link legs. The links are modeled as ellipsoids with a uniform density of mass, though preliminary calculations are performed using point masses. The motion is constrained to the 2-dimensional vertical sagittal plane; thus, we consider only forward, steady-state motion without lateral movement. The physical data used in our experiments can be found in Table 2.

The 22 continuous state and 8 control variables of the quadruped model are for legs $i = 1, \ldots, 4$:

$$\begin{array}{ll} x_1, x_2, x_3 & \text{torso orientation and position in the vertical plane} \\ x_4, x_5, x_6 & \text{torso angular and linear velocity} \\ x_{4i+3}, x_{4i+4} & \text{angle position and velocity of leg } i \text{ hip} \\ x_{4i+5}, x_{4i+6} & \text{angle position and velocity of leg } i \text{ knee} \\ u_{2i-1}, u_{2i} & \text{applied torque at leg } i \text{ hip and knee} \end{array} \quad (2.1)$$

A discrete state variable q_i is also associated with each leg and describes the contact situation of the i^{th} leg with the ground at time t:

$$q_i : [0, t_f] \to \{1, 2, 3\}, \; i = 1, \ldots, 4, \quad q_i(t) = \begin{cases} 1, & \text{no contact or swinging phase} \\ 2, & \text{fixed contact phase} \\ 3, & \text{slipping contact phase} \end{cases} \quad (2.2)$$

Here, we restrict ourselves to $q_i \in \{1, 2\}$. Thus, the number of possible discrete states at time t is $2^4 = 16$.

Figure 1 displays the quadruped hybrid automaton. The nodes contain the components of the discrete state for which $q_i = 2$, i.e. leg i is in contact state. Every edge represents a possible discrete transition. The principal quadruped walking objective is to traverse each of the four regions separated by the dashed lines so that each leg will have made contact once with the ground. The graph conveys the enormous discrete complexity of the problem.

At each node of the hybrid automaton, a continuous set of dynamics describe the state evolution. These equations are functions of the continuous states $x(t)$, discrete states $q(t)$, controls $u(t)$, parameters p, and time t,

$$\dot{x}(t) = \begin{cases} f^1(x(t), u(t), q(t), p, t), & t \in [t_0, t_{S,1}], \\ f^k(x(t), u(t), q(t), p, t), & t \in [t_{S,k-1}, t_{S,k}], \; k = 2, \ldots, m-1 \\ f^m(x(t), u(t), q(t), p, t), & t \in [t_{S,m-1}, t_f], \end{cases} \quad (2.3)$$

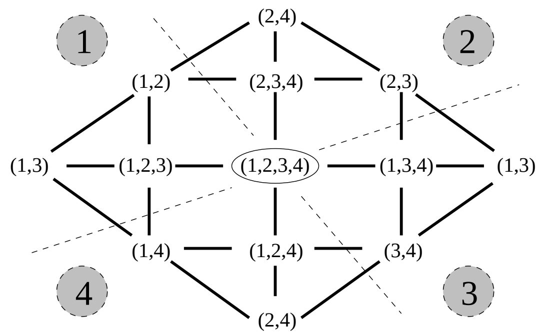


Figure 1: Hybrid automaton for the quadruped. The numbers in parentheses correspond to the support legs in that discrete state. Edges indicate discrete transitions.

where the discrete state $q(t)$ remains constant in each phase. The state equations of the biped walker may be obtained from the multibody dynamical equations experiencing contact forces,

$$\ddot{\theta} = \mathcal{M}^{-1}(u + J_c^T f_c - \mathcal{C} - \mathcal{G})\,. \tag{2.4}$$

In Equation (2.4), the state $x(t) = (\theta, \dot{\theta})$ contains the minimal generalized coordinates of the system as given by (2.1), $\mathcal{M}(\theta)$ is the square, positive-definite mass-inertia matrix, $\mathcal{C}(\theta, \dot{\theta})$ is the vector of Coriolis and centrifugal forces, $\mathcal{G}(\theta)$ is a vector of gravitational forces, u are the applied torques at the links, $J_c(\theta)$ is the constraint Jacobian, and f_c is the constraint force.

For computing the multibody equations of motion, we use recursive, symbolic multibody algorithms based on the work in (9). A further development of these algorithms together with special reduced dynamics algorithms intended for legged machines may be found described in (7). The reduced dynamics algorithms allow the equations of motion to be represented in a reduced form eliminating the contact equality constraints, thus converting the differential-algebraic system into an ODE system. A collision of a leg with the ground is modeled as an instantaneous, plastic collision. An impulsive force is thereby introduced to the contact legs which in turn produces a discontinuous jump in the generalized velocities. In addition, the unknown constant parameters p which influence the dynamics are: p_1 : *stride length (m)*, p_2 : *offset of opposing legs (m)*, p_3 : *average forward velocity (m/min)*. The time events $t_{S,k}$ at which the various phases occur are also unknown and are free to vary. Furthermore, the third parameter p_3 can be fixed at a desired value if we wish to find the optimal gait at a given forward velocity.

2.3 Hybrid optimal control

We are interested in gait patterns which minimize the injected energy per meter traveled required for periodic locomotion. This can be measured up to a proportionality constant as the squared integral of the applied torques divided by the length of the stride. We formulate this problem as the optimal hybrid control problem:

$$J(u) = \min_u \Big(\sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} u(t)^T u(t)\, dt \Big) / p_1 \tag{2.5}$$

subject to the nonlinear dynamics (2.3) and *location invariant* conditions:

$$(x(t), u(t)) \in Inv(q) \qquad x(t) \in X \subset \mathbb{R}^{n_x}, \qquad u(t) \in U \subset \mathbb{R}^{n_u}, \qquad q \in Q \subset \mathbb{Z}^{n_q} \tag{2.6}$$

where *Inv* is called the *location invariant* (13) of q. This function defines for a discrete state q the valid regions of the the continuous state space X and continuous control space U. The location invariant condition may also be expressed with inequality and equality contraints,

$$\begin{array}{rclll} g_i^k(x(t),u(t),q(t),p,t) & \geq & 0, \quad t \in [t_{S,k-1}, t_{S,k}], & i = 1,\ldots,n_{g_m^k}, & k = 1,\ldots,m, \\ h_i^k(x(t),u(t),q(t),p,t) & = & 0, \quad t \in [t_{S,k-1}, t_{S,k}], & i = 1,\ldots,n_{h_m^k}, & k = 1,\ldots,m. \end{array} \tag{2.7}$$

Additional constraints exist at the time events $t_{S,k}$ when a discrete transition $q \to q'$ takes place. In this case, the following functions must be satisfied:

$$\begin{array}{rcl} (x(t_{S,k}^-), u(t_{S,k}^-), x(t_{S,k}^+), u(t_{S,k}^+)) & \in & Guard_{q,q'} \\ x(t_{S,k}^+) & \in & Jump_{q,q'}(x(t_{S,k}^-), u(t_{S,k}^-)) \end{array} \tag{2.8}$$

where x and u evaluated at $(t_{S,k}^-)$ and $x(t_{S,k}^+)$ represent the values before and after the k^{th} time event respectively. The function *Guard* determines if a discrete state transition may take place, and the function *Jump* prescribes the jump in the continuous state as a result of a change in the discrete state. These functions may also be written as boundary constraints at the time events.

$$\begin{array}{rclrcl} r_i^1(x(t_0),u(t_0),q_0,p,0,x(t_f),u(t_f),q_f,t_f) & = & 0 & i & = & 1,\ldots,r_m^k \\ r_i^k(x(t_{S,k}^-),u(t_{S,k}^-),q(t_{S,k}^-),p,t_{S,k},x(t_{S,k}^+),u(t_{S,k}^+),q(t_{S,k}^+)) & = & 0 & k & = & 2,\ldots,m-1 \end{array} \tag{2.9}$$

We only consider here *autonomous switching* in which a discrete state transition is forced internally by the constraints on the problem rather than through a discrete control action.

In relation to the optimal gait problem, we make the following associations:
Inv The continous states x must satisfy kinematic constraints depending upon which legs are in contact with the ground. The contact forces must point upwards and the horizontal forces must be small enough such that it does not enter into a slipping state.
Guard A leg can break contact with the ground when the vertical contact force reaches zero which may happen on its own or as a result of collision with another leg with the ground. A leg makes contact with the ground when in the proper kinematic configuration.
Jump At a collision of a leg(s) with the ground, a discontinuous jump in the state velocities occurs as a result of an impulse force(s) propagating throughout the body.

3 NUMERICAL HYBRID OPTIMAL CONTROL

The numerical solution of general hybrid systems is still in its infancy. Precisely the unknown switching structure, i.e. the discrete state trajectory, of the control problem is what traditional numerical optimal control programs have difficulty handling. Such a framework can not be treated directly by gradient-based approaches nor with purely discrete optimization methods. In (3), algorithms were presented for the solution of optimal hybrid control problems, and more recent efforts for developing new algorithms may be found in (8, 12). The algorithms are characterized by a discretization of the state and control spaces so as to approximate the optimized cost function and thereby produce the optimal control actions. In practice, these approaches suffer severly from the well-known *curse of dimensionality* and would not be well-suited for solving for optimal quadruped gaits.

Here, we outline a different approach related to that in (4, 11). Assuming the existence of a lower bound on the length $t_{S,k+1} - t_{S,k}$ of a phase and, thus, excluding chattering there will only be a finite number of phases for a finite final time t_f. Furthermore, assuming a given number m of phases and q being constant in each phase, then each $q_i(t),\ 0 \le t \le t_f$ can be described by a vector of integers $z \in \mathbb{Z}^m$ with $q(t) = z_i$ in the i-th phase. Thus, the hybrid optimal control problem is transcribed into a *mixed-integer optimal control problem*. Furthermore, integer variables can generally be described by binary variables, thus obtaining a *mixed-binary optimal control problem* (4, 11). A Branch and Bound (B&B) technique is then applied to search the entire solution space by doing a truncated binary tree search for the discrete variables maintaining upper and lower bounds on the performance index. The binary variables are partially relaxed (allowed to vary between 0 and 1) at an inner node thus defining an optimal control problem with dynamic equations defined in multiple phases. Its solution provides a lower bound on the performance index for all nodes of the subtree. If the lower bound is greater than the current global upper bound then the entire subtree is fathomed.

We use sparse direct collocation (10) for solving for the optimal open-loop controls $u^*(t)$ to the multiphase optimal control problem. It is equipped to handle general nonlinear equality and inequality contraints on the states and controls including magnitude bounds, multiple phases with switching dynamics, jumps in the states and controls, and objectives with continuous and discrete costs. This program uses an optimization method based on the method of direct collocation which solves for the states and controls at an (unknown) sequence of time points $t_{S,j-1} = t_1^j < t_2^j < \cdots < t_{n_{G,j}}^j = t_{S,j}$,

$$\begin{array}{rcll} \tilde{u}_{\text{app}}(t) & = & \beta(\hat{u}(t_k^j), \hat{u}(t_k^{j+1})), & \beta - \text{linear} \\ \tilde{x}_{\text{app}}(t) & = & \alpha(\hat{x}(t_k^j), \hat{x}(t_k^{j+1}), f_k^j, f_{k+1}^j), & \alpha - \text{cubic} \end{array} \qquad t \in [t_k^j, t_{k+1}^j] \tag{3.1}$$

where $f_k^j = f^j(\hat{x}(t_k^j), \hat{u}(t_k^j), q^j, p, t_k^j)$. A finite-dimensional constrained nonlinear program is thereby obtained for a fixed value of q and the unknown values for x, u, p, E. The problem is then solved using an SQP-based optimization code for sparse systems SNOPT (6).

The numerical calculational approach for solving hybrid optimal controls described here is based on making *cuts* in the discrete search tree such that the discrete complexity becomes manageable. Since even with very few discrete states the number of potential paths may be enormous, a total enumeration of all possibilities is generally not feasible. For example, assume 4 discrete states with only 5 phases, then we have up to $(2^4)^5 = 1,048,576$ possible discrete state trajectories. However, in the case of the quadruped, this number can be reduced with additional assumptions such as symmetry and restricting the possible discrete transitions.

4 NUMERICAL OPTIMAL WALKING – EXPERIMENTS AND STRATEGY

4.1 Numerical experiments

It is to be expected that for our quadruped model, different walking patterns will be optimal at different desired walking speeds. In (1), a study was presented in which for different four-legged animals, a wide variation of gait patterns existed depending on average forward velocity. A horse, for example, will first amble, then trot, canter, and finally gallop. In our experiments, we study the energy requirements for the optimal walk of two different gait patterns: the amble and the trot.

We first solved for the minimum energy open-loop controls at the eight hip and knee koints with respect to a point mass model of our quadruped. Using symmetry constraints, we were

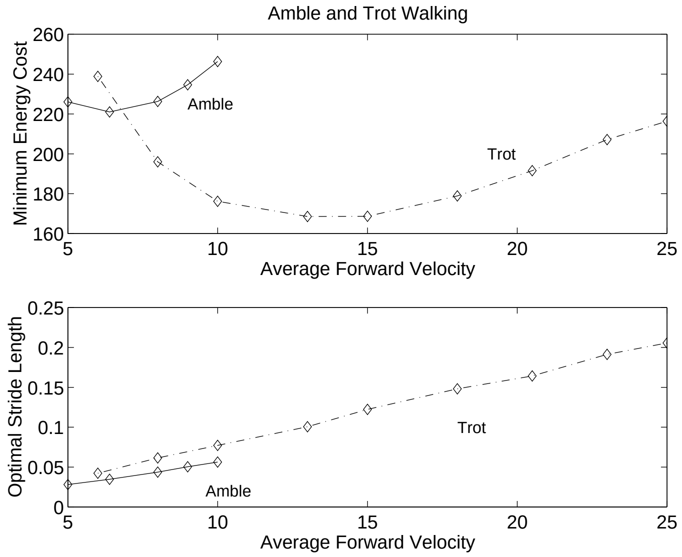


Figure 2: Minimum energy plots and optimal strides for amble and trot walking gaits.

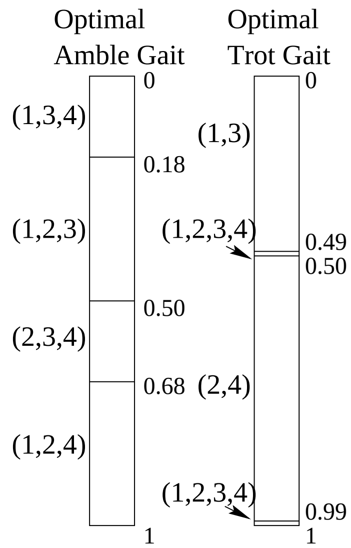


Figure 3: Relative phases of the optimal amble and trot gaits. Numbers in parentheses indicate the support legs.

able to formulate the problem as half a stride. Each leg moves individually and the next leg breaks contact at the same moment as the previous leg collides with the ground. As expected the optimum forward speed lies at the relatively low velocity of 6.39 *m/sec*. See Figure 2. The optimal stride lengths are as expected monotonically increasing with increased speed.

The trot gait, on the other hand, is characterized by two diagonally opposed legs swinging simultaneously. After collision, a short full contact phase is modeled where all legs are in contact with the ground. Figure 2 displays how this gait is noticeably more efficient at higher speeds having an optimal forward velocity of 13.0 *m/sec*, though there exists a point slow enough at 7.0 *m/sec* for which amble becomes more efficient. Figure 3 displays the optimal relative phases for both the amble and trot gaits at their optimal velocity. The amble relative phases differ from Table 1, most likely due to the particular construction of our quadruped, see Table 2. Figure 3 also illustrates how the optimal hybrid trajectories of each gait pattern correspond to the hybrid automaton presented in Figure 1. The numbers in parentheses give the numbers of the support legs and correspond to the current hybrid state of the system. These experimental results present two radically different hybrid trajectories with different discrete transitions.

4.2 Solving for global solutions of quadruped walking.

Though we have solved for the optimal hybrid trajectories within two gait classes having optimized over the relative phases and the duty factors of the legs, what remains is to search over all gait classifications. To this end, work is in progress to implement the scheme described in Section 3 building upon the work in (11). A relaxation of the discrete state variables in the discrete search tree will result in the modeling of a continuous range between contact and a free swinging leg. Physically, this can be interpreted as a soft ground with a spring-force contact condition and a variable friction coefficient. This approach cannot search over all gait classifications and may not even include both the trot and amble considered here. This local optimal hybrid search would serve, however, to significantly reduce the discrete complexity. Neighboring solutions which may include a full leg contact phase in between swinging phases can then simultaneously be considered with those which do not. A remaining enumeration of the other regions of the discrete state space would then be possible.

5 CONCLUSION

Our investigation into the generation of minimum energy symmetric, periodic gaits gathers together several different research areas in the modeling and control of complex, nonlinear and hybrid systems. Our ability to solve this problem has relied upon the use of recursive, symbolic multibody algorithms coupled with powerful numerical optimal control software. We frame the problem of finding the best gait for a quadruped for a given velocity as a hybrid optimal control problem. To a certain extent, we are already able to solve this problem. Our preliminary results demonstrate the power and potential of these numerical methods. The complete problem is a challenging problem in hybrid optimal control, and this work represents a first step in the direction of providing efficient numerical tools for handling these problems.

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