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Towards Hybrid Optimal Control

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Abstract: In this article a general class of hybrid optimal control problems with continuous and discrete state variables and control inputs is defined. After a brief review of conventional optimal control, major novel challenges resulting from the hybrid nature are discussed. Some application problems are comparatively easy to solve because of the fixed or known sequence of discrete events; however, if the number and the sequence of discrete phases is not known a priori, the solution must then be found among a combinatorial number of possible sequence candidates. The article presents several preliminary approaches to the (numerical) solution of hybrid optimal control problems by hybrid dynamic programming, by decomposition using branch-and-bound, or fixing transversality conditions to obtain suboptimal solutions. The last two methods rely on the capabilities of the direct collocation method DIRCOL to solving multi-phase optimal control problems robustly and efficiently. Results obtained by the proposed methods are presented in two examples: an underactuated robotic system with a holding brake as the discrete component, and a hybrid, motorized traveling salesman problem.

1 Introduction

The purpose of this article is to provide an overview of current research in theoretical and numerical methods for solving hybrid optimal control problems, i.e., optimal control problems with continuous and discrete state and control variables. Preliminary results concerning system theoretical issues connected to mathematical modeling and to the formal definition of hybrid optimal control problems are presented as well as first numerical approaches for solving fairly general hybrid optimal control problems.

Conventional optimal control problems with continuous state and control variables are commonly presented as optimization problems in the form of two- or multi-point boundary value problems (TPBVP or MPBVP) [25]. Studies for this class have extended to those problems containing inequality constraints on the control and state variables [14] and also to multi-phase problems with discontinuities in the system equations at intermediate time points [7, 9]. Intermediate time points at which the system equations suddenly change are denoted as *switching times*, and they separate the problem into phases each with its own dynamics. Although the times of switching may be unknown, a common and critically simplifying assumption is that the sequence of the phases, i.e., the switching structure, is more or less known.

It is not generally the case, that the switching structure is known for the class of hybrid dynamical systems formed by a set of continuous time control systems and a finite automaton that drives the switchings between different system structures [4, 24, 29]. As will become clear in the following, the right hand sides of the nonlinear differential(-algebraic) equations are of variable structure and may vary among a discrete set of combinatorial choices. When calculating the optimal control of the hybrid dynamical system, not only do the optimal trajectories of the continuous state and control variables and the unknown switching points between phases have to be determined but also the optimal sequence of the phases, i.e., the discrete state trajectory. It is the intrinsic combinatorial complexity, in addition to the nonlinearity of the continuous optimal control problem that forms the challenges in the theoretical and numerical solution of hybrid optimal control problems.

Some typical mechatronic application examples of hybrid optimal control problems are presented such as an underactuated robot and a motorized traveling salesman. One of the research goals in hybrid optimal control in the context of a multilegged walking machine is to find the optimal solution for the ground contact positions of each leg, the body and control trajectories, and simultaneously solve the combinatorial aspect of the problem by finding the optimal gait and number of steps. The resulting hybrid trajectory must be optimal, e.g., with respect to the overall power consumption of the machine.

For the organization of the article: In Section 2 a general class of hybrid optimal control problems is defined and illustrated in practical application examples. Section 3 discusses some theoretical issues resulting from the combinatorial complexity of choices for optimal discrete trajectories in the context of switched phase transitions of optimal control problems. Analytical solutions even to most example problems cannot be obtained. Thus, for solving practical problems numerical methods are needed to efficiently compute approximate solutions. The direct collocation method DIRCOL which provides highly efficient solutions of nonlinearly constrained multi-phase optimal control problems is briefly reviewed in Section 4. Subsequently, two general approaches to the numerical solution of hybrid optimal control problems are presented. The first method searches for a global optimal solution by a branch-and-bound technique, while the second one aims at a local, suboptimal solution by fixing transition times and states to certain values. Two application example problems solved by the proposed numerical methods are presented in Section 5. Section 6 summarizes the article and mentions possible directions for further research in the area of hybrid optimal control.

2 Hybrid Optimal Control

The discrete-continuous process model of a hybrid optimal control problem consists of a set of ordinary differential or differential-algebraic equations of variable structure and variable constraint equations. The system structure varies among a (finite) discrete set of system descriptions each of which is associated with a specific discrete state of the considered hybrid system. The discrete state dynamics may be modeled, e.g., by a finite state automaton or a Petri-net. For some modeling paradigms of hybrid dynamical systems see, e.g., [4, 10, 17, 21, 26].

Most of the hybrid models in the literature consider the hybrid state of the system as a combination of the continuous state $\mathbf{x}$ and a discrete state $\mathbf{q}$. Likewise, the control input is a combination of a continuous component $\mathbf{u}$ and a discrete-valued component $\mathbf{v}$. The hybrid system state and structure changes discontinuously when an autonomous or controlled discrete event at a particular time or state occurs.

The hybrid optimal control problem is to find optimal hybrid — i.e., continuous $\mathbf{u}$ and discrete $\mathbf{v}$ — control trajectories such that an integral cost index — typically an integral of a function of the hybrid system state and control input — is minimized subject to the system dynamics, initial, terminal and further equality or inequality constraints. A general class of hybrid optimal control problems is defined in the following.

Definition 1 *The hybrid optimal control problem is defined as the minimization of the hybrid cost index J*

$$\min_{\mathbf{u},\ \mathbf{v}} J(\mathbf{u},\mathbf{v}) = \Theta + \int_{t_a}^{t_e} \psi(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t)\ \mathrm{d}t\ , \qquad (1)$$

subject to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t) \qquad \text{if } s_j(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t) \neq 0 \quad (2)$$
$$j = 1,\ldots,n_s$$

$$\begin{bmatrix}\mathbf{x}(t_i^+)\\ \mathbf{q}(t_i^+)\end{bmatrix} = \phi_j(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t_i^-) \quad \text{if } s_j(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t_i^-) = 0 \quad (3)$$
$$j \in \{1,\ldots,n_s\}$$

$$\mathbf{u}(t) \in \mathcal{U} \subset \mathbb{R}^{n_u},\ \mathbf{v}(t) \in \mathcal{V} \subset \mathbb{Z}^{n_v},$$
$$\mathbf{x}(t) \in \mathcal{X} \subset \mathbb{R}^{n_x},\ \mathbf{q}(t) \in \mathcal{Q} \subset \mathbb{Z}^{n_q}, \quad \forall t \in [t_a,t_e] \qquad (4)$$

$0 \leq \mathbf{h}(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t), t \in [t_a,t_e]$ *inequality constraints,* (5)

$\mathbf{x}(t_a) = \mathbf{x}_a,\ \mathbf{q}(t_a) = \mathbf{q}_a$ *initial conditions,* (6)

$\mathbf{x}(t_e) = \mathbf{x}_e,\ \mathbf{q}(t_e) = \mathbf{q}_e$ *terminal conditions,* (7)

where the initial and final times t_a, t_e are free or fixed, s_j are the n_s switching functions and ϕ_j denotes the explicit phase transition conditions (jump maps) occuring at the zeros of one of the switching functions. The Mayer type part Θ of the performance index is a general function of the phase transition times (events) t_i, $i = 0,\ldots,N$, of the continuous $\mathbf{x}(t_i^-)$, $\mathbf{x}(t_i^+)$ and discrete states $\mathbf{q}(t_i^-)$, $\mathbf{q}(t_i^+)$ just before and just after the transition event written as

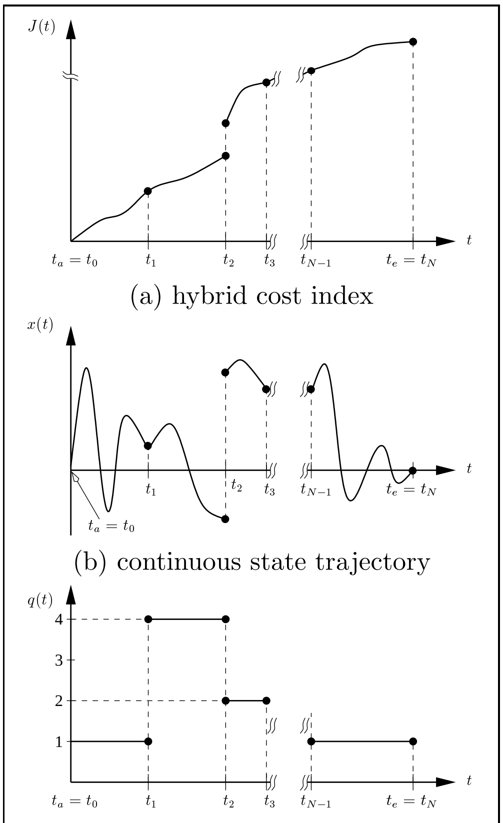
$$\Theta := \Theta[\,\mathbf{x}(t_0^-),\mathbf{x}(t_0^+),\ldots,\mathbf{x}(t_N^-),\mathbf{x}(t_N^+);\ \mathbf{q}(t_0^-),\mathbf{q}(t_0^+),\ldots,\mathbf{q}(t_N^-),\mathbf{q}(t_N^+);t_0,\ldots,t_N]\ .$$

Here, $t_a = t_0$, $t_e = t_N$ is assumed while the number of phases N may be given or free. The integrand ψ is a real-valued function of the continuous/discrete state and control variables and of time.

The minimization of (1) is subject to the initial and terminal conditions (6), (7), admissible values for the continuous/discrete control variables (4), and inequality constraints (5). Obviously, valid hybrid optimal trajectories have to obey the differential equations (2) and the phase transition equations (3) of the discrete aspect. The optimization parameters to be determined are the continuous $\mathbf{u}(t)$ and discrete control input trajectories $\mathbf{v}(t)$ and all, some, or none of the phase transition times.

Figure 1 shows typical trajectories of a solution to a hybrid optimal control problem. The cost index is shown in Figure 1(a), where a discontinuous jump in the cost occurs at the transition time t_2. Likewise, the continuous state trajectory $\mathbf{x}(t)$ shown in Figure 1(b) may have discontinuities in the state at time t_2 and in its rate at time t_1. The discrete state trajectory $\mathbf{q}(t)$ is shown in Figure 1(c).

Remark 1 An integer-valued, scalar discrete state $q \in \mathcal{Q} = [q_{\min}, q_{\max}]$ can be transformed to an n_w-dimensional binary-valued discrete state vector



(a) hybrid cost index

(b) continuous state trajectory

(c) discrete state trajectory

Figure 1: Typical trajectories of a hybrid system.

$\mathbf{w} \in \{0,1\}^{n_w}$ by

$$q = q_{\min} + w_1 + 2w_2 + 4w_3 + \ldots + 2^{n_w - 1} w_{n_w},$$

with $n_w = 1 +$ INT $\{\log{(q_{\max} - q_{\min})}/\log 2\}$. In this case, the hybrid optimal control problem of Definition 1 can be recasted using binary instead of integer variables.

The solutions to hybrid optimal control problems presented in Definition 1 are deterministic open-loop trajectories. Like in conventional optimal control this problem class can be generalized to a stochastic setting or to issues like optimal closed-loop feedback control. Currently, mainly the open-loop problem is discussed in the literature with many open issues. Further research is required to define and understand stochastic and closed-loop hybrid optimal control.

Example 1 An example of a hybrid mechatronic system is the underactuated robot R2D1 with 2 rotational degrees-of-freedom (DOFs) and only 1 actuator. Figure 2 shows the kinematic structure of R2D1 with the torque u_1 in the 1st joint and a holding brake in the 2nd joint. The plane of the SCARA robot can be inclined versus gravitation. One of the typical control problems in underactuated systems is the positioning of the unactuated joint, usually resulting in unstable zero dynamics. For details and hybrid (switching) globally stable control of R2D1 see [19, 20]; an overview of control strategies for underactuated (robot locomotion) systems is given in [27].

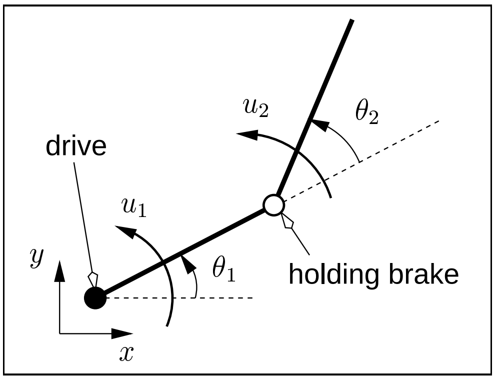


Figure 2: Kinematic structure of R2D1.

Here, we consider the hybrid optimal control problem defined by the task to bring R2D1 from a given initial state to a desired goal state. This is a hybrid (multiphase) problem because it is not known a priori how often, at what times, and at which states the holding brake should be activated.

The formal problem statement is as follows

$$\min_{u,v} J = \min_{u,v} \int_0^{t_e} \left((\mathbf{x} - \mathbf{x}^d)^T \mathbf{W} (\mathbf{x} - \mathbf{x}^d) + \alpha u_1^2 \right) \, \mathrm{d}t \tag{8}$$

subject to the robot dynamics

$$\mathbf{M}\ddot{\theta} + \mathbf{h}(\theta, \dot{\theta}) = \begin{bmatrix} u_1 \\ u_2 = \begin{cases} 0 & \text{if } q = 1 \\ u_K & \text{if } q = 2 \end{cases} \end{bmatrix} \tag{9}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \quad q(0) = 1 \tag{10}$$

$$\mathbf{x}(t_e) = \mathbf{x}_e \quad q(t_e) = 2 \; , \tag{11}$$

where $\mathbf{W} \geq 0$, $\alpha > 0$, with $\mathbf{x}^d(t)$ a desired reference trajectory. The continuous state vector $\mathbf{x} = [\theta_1 \; \dot{\theta}_1 \; \theta_2 \; \dot{\theta}_2]^T$ consists of the two joint angles and angular velocities. The discrete state q corresponds to the on/off state of the holding brake, which is assumed to be directly controlled by a discrete (on/off) control input v; u_K is the breaking torque. Discontinuity surfaces s_j and transition maps ϕ_j according to (2), (3) are defined in the obvious way with no discontinuities in the continuous state, i.e., $\mathbf{x}(t_i^+) = \mathbf{x}(t_i^-)$ for all event times t_i.

Solutions to this problem are presented later in Section 5.1

3 Theoretical Issues

3.1 Continuous optimal control problems

There are two basic approaches to solving conventional optimal control problems:

(I) Hamilton-Jacobi-Carathéodory-Bellman (HJCB) partial differential equations (PDEs) and dynamic programming, and

(II) classical Variational Calculus, Euler-Lagrange differential equations (EL-DEQ), and the Maximum Principle.

The first approach theoretically provides optimal *feedback* controls $\mathbf{u}^*(\mathbf{x}, t)$ by solving the PDE for the value function

$$-\frac{\partial V(\mathbf{x},t)}{\partial t} = \min_{\mathbf{u}(t)\in\mathcal{U}} \left\{ \psi(\mathbf{x},\mathbf{u},t) + \left(\frac{\partial V(\mathbf{x},t)}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x},\mathbf{u},t) \right\}$$

with $V(\mathbf{x}, t_e) = \Theta(\mathbf{x}, t_e)$. However in practice, the PDE can only be solved numerically for very small state dimensions. The same holds for the corresponding dynamic programming method which suffers from the curse of dimensionality. A further, severe drawback is that inequality constraints on the state variables as well as switched dynamical systems usually lead to discontinuous partial derivatives of V and cannot easily be included. In this context, it is rarely known that Bellman's results have been preceded by results of Carathéodory [23].

For the second approach, the mentioned drawbacks usually do not hold, but only an optimal *open loop* control $\mathbf{u}^*(t)$ is obtained together with the trajectories of the optimal state $\mathbf{x}^*(t)$ and adjoint (or co-state) variable $\lambda^*(t)$. This is done by solving the EL-DEQ in $[t_a, t_e]$

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \lambda} = \mathbf{f}(\mathbf{x},\mathbf{u},t), \quad -\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial \mathbf{x}} = \lambda^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \psi}{\partial \mathbf{x}}$$

while the optimal control minimizes the Hamiltonian

$$\mathcal{H}(\mathbf{x},\mathbf{u}^*,\lambda,t) = \min_{\mathbf{u}(t)\in\mathcal{U}} \left\{ \lambda^T \mathbf{f}(\mathbf{x},\mathbf{u},t) + \psi(\mathbf{x},\mathbf{u},t) \right\}$$

along the optimal trajectory $\mathbf{x}^*(t)$, $\lambda^*(t)$. The first and second approach are related by $\lambda^*(t) = \partial V(\mathbf{x}^*(t), t)/\partial \mathbf{x}$ which holds along unconstrained sections of the optimal trajectory. The solutions of the EL-DEQ describe characteristics of the HJCB-PDE. This approach has also been extended to handle general constraints on the control and state variables [14]. Then, the EL-DEQ form a MPBVP with a priori unknown interior switching points denoting the times when one of the constraints becomes active or inactive [8]. Activation or deactivation of a state constraint switches the adjoint differential equations.

In recent years, research has also been focused on a numerical synthesis of optimal feedback controls from a family of very many characteristics of the HJCB PDE (i.e., optimal open loop trajectories) using local Taylor sums [5] or neural networks [6].

3.2 Multi-phase systems with given order of phases

In many application fields, the order of phases is known (or given) a priori, e.g., the trajectory optimization of a hypersonic two-stage space vehicle in flight mechanics [9]. The switching between two subsequent phases is usually handled with discontinuities in the state variables at interior points [7]. For this subset of hybrid optimal control problems where the sequence of phases is given, a selection of combinatorial complexity among several alternative phases is not required at each phase transition. Nevertheless, such multi-phase problems usually cannot be solved analytically, except for possibly in the case of very simple dynamical equations. Problems of practical relevance must therefore be solved numerically, e.g., using the direct collocation method DIRCOL discussed in Section 4.2.

3.3 Unknown sequence of phases and transitions

In generic hybrid optimal control problems, usually neither the order of phases, the times (events) of phase transitions (switchings) nor even their number are known in advance. These need to be determined as part of the optimal solution which usually results in a significant combinatorial complexity. In principle, a multi-phase optimal control problem with unknown phase transition times and states is associated with each of the discrete state sequence candidates. A straightforward approach to obtain the optimal solution is to solve all the multi-phase problems for all possible discrete state sequences and to select the best one. However, the number of possible sequences may be NP-complete or even infinite when allowing discrete states to cycle. Thus, even for a moderate number of phases this approach is not practically feasible.

The new key challenge when solving hybrid optimal control problems is to reduce the number of discrete state sequence iterates and therewith the number of multiphase solutions needed. There are various ways to perform the search among all possibilities, e.g., branch-and-bound as discussed in Section 4.3, heuristics, or other simplifying assumptions, all in search for practical and possibly suboptimal solutions to the hybrid problem.

A hybrid version of the Pontryagin Maximum Principle as well as of the HJCB-PDE are further issues of theoretical interest. Hybrid generalizations of optimality conditions — e.g., the choice of the controls such that the Hamiltonian is minimized at all times — could provide powerful analytical and numerical methods to solving hybrid optimal control problems. Such theoretical questions are currently open issues; some preliminary results can be found in recent publications, e.g., [15, 24, 29].

The determination of the unknown switching structure of active state and control constraints in conventional optimal control may be viewed as a related problem. In this context homotopy or continuation techniques have been applied successfully [22]: A simplified problem, e.g., with relaxed or untightened or even no constraints, is solved first. Starting from this first solution a second problem is solved with less simplifications, e.g., with a first active constraint and a first, yet simple switching structure, e.g., a touch point, at the solution. This procedure is continued until one ends up with a solution to the fully constrained problem possibly exhibiting a complicated switching structure.

3.4 Hybrid dynamic programming

The principle of optimality and the resulting method of dynamic programming may be generalized to hybrid problems, see e.g., [16] for a proposal of a hybrid Hamilton-Jacobi-Bellman formulation using chattering approximations.

As a consequence, hybrid optimal control problems may in principle be solved by a hybrid version of the method of dynamic programming. This is illustrated in an example with simple (unique) discrete dynamics.

Example 2 Let us assume a hybrid system with two discrete states $q = 1,\ q = 2$, with the initial condition being $q = 1$, $\mathbf{x}(t_a) = \mathbf{x}_a$ and the desired terminal state of $q = 2$, $\mathbf{x}(t_e) = \mathbf{x}_e$, and free terminal time t_e. The continuous dynamics are

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}(\mathbf{x}, \mathbf{u}, q = 1) = \mathbf{f}_1(\mathbf{x}, \mathbf{u}), \\ \mathbf{f}(\mathbf{x}, \mathbf{u}, q = 2) = \mathbf{f}_2(\mathbf{x}, \mathbf{u}) \ . \end{cases}$$

Furthermore, it is assumed that exactly one phase transition occurs at the unknown time t_1 and state $\mathbf{x}(t_1)$ satisfying the interior point constraint $\tilde{\mathbf{g}}(\mathbf{x}(t_1), t_1) = 0$. The discrete state sequence of the example is $q = 1 \overset{t_1}{\rightsquigarrow} 2$. Optimal trajectories $\mathbf{u}^*(t)$, $\mathbf{x}^*(t)$, switching time t_1^*, switching state $\mathbf{x}^*(t_1)$ with respect to the cost index $J(\mathbf{x}, \mathbf{u}, t_e)$ are to be determined.

This two phase problem may be solved using hybrid dynamic programming, see Figure 3. Beginning with the terminal state the TPBVP $\min\limits_u J(\mathbf{x}, \mathbf{u}, t_e)$ of phase 2 is solved subject to $\mathbf{x}(t_e) = \mathbf{x}_e$ and the free (to be determined) interior point time t_1 and state $\mathbf{x}(t_1)$, satisfying $\tilde{\mathbf{g}}_1 = \mathbf{0}$. The optimal solution is to be parameterized as $\mathbf{u}^*(\mathbf{x}(t_1), t_1)$ and the resulting remaining cost for phase 2 also follows as a function of interior point time and state as $V(\mathbf{x}(t_1), t_1) = \left[\min\limits_u J\right]_{t_1}^{t_e}$. After the remaining optimal cost of phase 2 is available[1], we can solve for phase 1 by the principle of optimality as another TPBVP with initial condition $\mathbf{x}(t_a) = \mathbf{x}_a$ and terminal constraint $\tilde{\mathbf{g}}_1$ and combine this solution with the remaining cost $V(\mathbf{x}(t_1), t_1)$ for phase 2 to obtain the overall solution $V(\mathbf{x}(t_a), t_a)$ to the two phase problem as

$$V(\mathbf{x}(t_a), t_a) = \left[\min_u J\right]_{t_a}^{t_e} = V(\mathbf{x}(t_1), t_1) + \left[\min_u J\right]_{t_a}^{t_1}$$

Clearly, as illustrated in the example, the complexity of the expressions for the remaining cost parameterized by

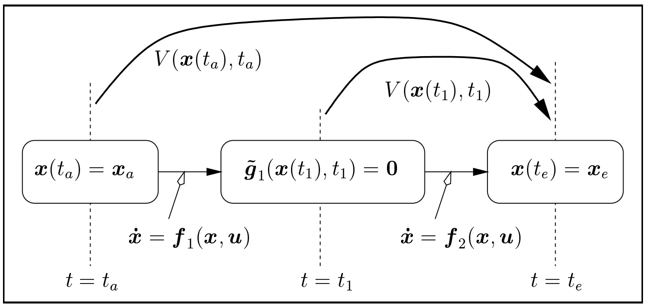


Figure 3: Example of a hybrid optimal control problem with unique discrete state sequence.

the interior point times t_i and states $\mathbf{x}(t_i)$ increases as the initial condition of the hybrid optimal control problem is approached from the terminal constraint with each backwards step of dynamic programming. The difficulty in solving for the remaining cost further increases if there exist several choices for the discrete state sequence as shown in Figure 4(a). One of the benefits of a hybrid version of dynamic programming is that the complexity of the overall problem may be reduced whenever two paths backwards through the graph (with nodes representing the interior points) join, see Figure 4(b). Dynamic programming in this case can deliver a strategy of which path to choose; however, the synchronisation problem of the times on different paths and the forward reachability problem is another complication.

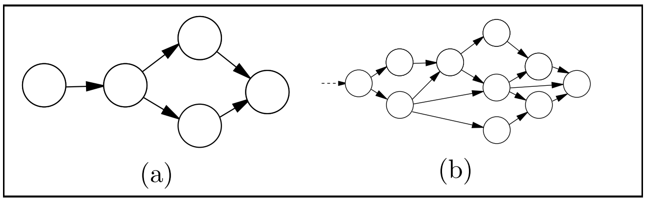


Figure 4: Further examples of discrete state sequences.

To summarize this discussion about hybrid dynamic programming: For problems, where it is possible to join the various paths of the graph representing the discrete dynamics, the procedure for choosing the optimal path may be simplified. Because the remaining cost needs to be calculated with respect to the interior point time and state parameters, the cost expressions become all the more complicated for multi-phase problems, and the method of hybrid dynamic programming becomes much more difficult to apply.

These insights have inspired the development of a suboptimal method of hybrid dynamic programming presented in Section 4.4.

4 Numerical Issues

In this section numerical methods to efficiently solving nonlinearly constrained optimal control problems with multiple phases are briefly reviewed. The recently developed method DIRCOL based on sparse direct collocation is described in detail. This highly efficient numerical method makes a decomposition approach to hybrid optimal control problems of moderate discrete complexity

[1] Note that only in very simple cases of the continuous dynamical equations is it possible to analytically compute the optimal control and remaining cost as a function of interior point time and state.

possible: In an inner loop multi-phase optimal control problems are solved numerically for a given sequence of phases whereas the order and type of phases are varied during the outer iteration. Furthermore, we present an even faster but only suboptimal solution approach to a particular class of hybrid optimal control problems.

4.1 Numerical optimal control methods

Within the last thirty years, the solution approach described in Section 3.1 based on the EL-DEQ and the Maximum Principle has produced a rich and steady source of families of numerical methods of continuously increased efficiency which can mainly be divided into two classes: *direct* and *indirect* methods [33].

Indirect methods approximate a solution by the necessary conditions of optimality resulting from the Maximum Principle and the EL-DEQ. Among the first family of indirect methods are gradient methods [35] that are based on an iterative improvement of a control approximation by minimization of the Hamiltonian. In each iteration step, the equations of motion (2) are numerically integrated forwards while the adjoint differential equations are integrated backwards. Multiple shooting [8, 28] is among the most powerful numerical methods for solving the resulting MPBVP derived from the necessary conditions of optimality of a constrained nonlinear optimal control problem. Thus, a highly accurate and verified (with respect to necessary conditions of optimality) solution can be obtained.

Practical drawbacks of indirect methods are:

- Proper formulations of the necessary conditions (EL-DEQ etc.) must be derived.
- Suitable initial guesses of the state and adjoint trajectories must be provided to start the iterative methods.
- In order to handle active constraints properly, their switching structure must be guessed.
- Changes in the problem formulation (e.g., by a modification of the equations of motion), or low differentiability properties of the model functions (e.g., by low order interpolation of tabular data), cannot easily be included in the solution procedure.

Most of these drawbacks have been overcome by direct methods mainly developed during the last decade [3, 33] and pushed by the tremendous progress in nonlinear optimization methods [2, 3]. These methods are based on a transcription of optimal control problems into (finite dimensional) nonlinearly constrained optimization problems (NLPs) by a parameterization of the control variable u. Two different transcription strategies exist:

(i) Iterative simulation and optimization (direct shooting): In every iteration step of the optimization method, the equations of motion (2) are solved by a numerical integration method for the current guess of parameters.

(ii) Simultaneous simulation and optimization (direct collocation): The differential equations (2) are only fulfilled at a priori selected points using collocation as an implicit integration scheme. This leads to a system of nonlinear equality constraints for the parameters of the resulting NLP.

While (i) satisfies the equations of motion (2) in each iteration step, (ii) only satisfies them at a successful termination of the optimization procedure if a sequential quadratic programming (SQP) method is used [2]. For approach (i) the gradient information must be computed by numerical sensitivity analysis of initial value problems, while the gradient computation is easier for (ii). The number of NLP variables of approach (i) is usually much smaller than for approach (ii) where the number of variables is of the order of $(n_x + n_u)$ times the number of collocation points. On the other hand, NLP gradient and Jacobian structure of (ii) are very sparse.

4.2 Direct collocation method DIRCOL for multi-phase optimal control problems

Direct methods promise high flexibility and robustness when solving optimal control problems numerically to low or moderate accuracies. Additionally appealing in the direct collocation approach is the potentially faster computation compared to direct shooting. This is due to the simultaneous simulation and optimization approach and will only be effective if the NLP sparsity can fully be utitilized. Otherwise the NLP size will severly limit the efficiency.

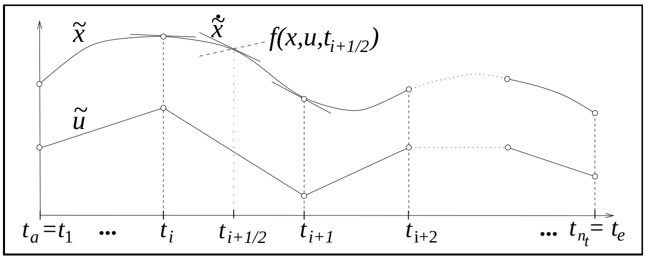


Figure 5: Direct collocation parameterization of continuous state and control variables.

In DIRCOL [31] a discretization of $\mathbf{x}$ by piecewise cubic Hermite polynomials $\tilde{\mathbf{x}}(t) = \sum_k \alpha_k \hat{\mathbf{x}}_k(t)$ and of $\mathbf{u}$ by piecewise linear functions $\tilde{\mathbf{u}}(t) = \sum_k \beta_k \hat{\mathbf{u}}_k(t)$ is applied [13, 30] on a discretization grid $t_a = t_1 < t_2 < \ldots < t_{n_t} = t_e$, see Figure 5. The equations of motions (2) are pointwise fulfilled at the grid points and at their respective midpoints resulting in a set of nonlinear NLP equality constraints $\mathbf{a}(\mathbf{y}) = 0$ (collocation). Any control or state variable inequality constraints are to be satisfied at the grid points resulting in set of nonlinear NLP inequality constraints $\mathbf{b}(\mathbf{y}) \geq 0$. Here, $\mathbf{y}$ denotes the n_y parameters of the parameterization

$$\mathbf{y} = (\alpha_1, \alpha_2, ..., \beta_1, \beta_2, ..., t_e)^T.$$

The resulting nonlinearly constrained optimization problem basically reads as

$$\text{NLP:} \quad \min_{\mathbf{y}} \varphi(\mathbf{y}) \quad \text{subject to} \quad \mathbf{a}(\mathbf{y}) = 0,\ \mathbf{b}(\mathbf{y}) \geq 0,$$

where φ denotes the parameterized cost index (1).

A carefully selected discretization $\tilde{\mathbf{u}}$, $\tilde{\mathbf{x}}$ must satisfy certain convergence properties. One requirement is that the discretized solution must approximate a solution of the EL-DEQ and the Maximum Principle if the grid becomes fine enough, i.e., for $n_t \to \infty$ and $\max\{t_{i+1} - t_i : i = 1, \dots, n_t - 1\} \to 0$ [30].

A great advantage of the direct collocation approach is that it provides reliable estimates $\tilde{\lambda}$ of the adjoint variable trajectory along the discretization grid. These estimates are derived from the Lagrange multipliers of the NLP [30]. They enable a verification of optimality conditions of the discretized solution although the EL-DEQ have not been solved explicitly.

Also, local optimality error estimates can be derived that enable efficient strategies for successively refining a first solution on a coarse grid [30, 31]. Thus, a sequence of related NLPs must be solved whose dimensions increase with the number of grid points.

NLPs can be solved most efficiently numerically by SQP methods. In each SQP iteration a current guess of the solution $\mathbf{y}^*$ is improved by the solution of a quadratic subproblem derived from a quadratic approximation of the Lagrangian of the NLP subject to the linearized constraints [2, 11]. The NLPs resulting from a direct collocation discretization have several special properties [32]:

- The NLPs are of large-scale with very many variables and very many constraints.
- Most of the NLP constraints are active at the solution, e.g., the equality constraints from collocation. Thus, the number of free NLP variables is much smaller than the total number of variables n_y.
- The NLP Jacobians $(\nabla \mathbf{a}(\mathbf{y}), \nabla \mathbf{b}(\mathbf{y}))$ are sparse and structured. Only a few percent of the elements will be nonzero, and the percentage decreases as the number of grid points increases.

These features can fully be utilized by the recently developed large-scale SQP method SNOPT [11]. The computational speedup achievable by utilizing the NLP structure is more than a factor of one hundred for typical discretized optimal control problems when compared to standard "dense" SQP methods [32].

To tackle fairly general multi-phase optimal control problems the recent version of DIRCOL [31, 32] has been extended to handle also

- dynamical equations defined in multiple phases,
- general phase connecting and phase transition conditions,
- phase dependent general inequality and equality constraints, i.e., differential-algebraic equations of variable structure.

4.3 Decomposition of hybrid optimal control problems using branch-and-bound

If the discrete state is identified with a finite sequence of phases and the discrete control can be described by an integer variable, then the hybrid optimal control problem can be described as a mixed-integer optimal control problem (MIOCP). A direct collocation or direct shooting discretization then results in a mixed-integer NLP (MINLP). The numerical solution of MINLPs is a topic of current research [1, 12]. Not many methods are available yet to handle nonconvex problems as is the case for discretized optimal control problems.

We suggest a decomposition approach to the solution of MIOCPs with multiple phases [32, 34]:

- In the inner iteration, a primal multi-phase optimal control problem is solved for given discrete state and control variables. This yields an upper bound on the hybrid performance index.
- In the outer iteration, the discrete variable is altered depending on global lower and upper bounds on the hybrid performance index.

More precisely, if the discrete states $\mathbf{q}$ are directly controlled by the discrete controls $\mathbf{v}$ (as in Example 1), and after the integer-valued variables have been represented by binary-valued variables $\mathbf{w} \in \{0,1\}^{n_w}$ (cf. Remark 1), the MIOCP reads as follows:

Definition 2 *The mixed-integer (mixed-binary) optimal control problem is defined as the minimization of the hybrid cost index J*

$$\min_{\mathbf{u},\ \mathbf{w}} J(\mathbf{u}, \mathbf{w}) = \Theta + \int_{t_a}^{t_e} \psi(\mathbf{x}, \mathbf{u}, \mathbf{w}, t)\ \mathrm{d}t\ , \qquad (12)$$

subject to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}, t) \qquad \text{if} \quad s_j(\mathbf{x}, \mathbf{u}, \mathbf{w}, t) \neq 0 \qquad (13)$$

$$\begin{bmatrix} \mathbf{x}(t_i^+) \\ \mathbf{q}(t_i^+) \end{bmatrix} = \phi_j(\mathbf{x}, \mathbf{u}, \mathbf{w}, t_i^-) \quad \text{if} \quad s_j(\mathbf{x}, \mathbf{u}, \mathbf{w}, t_i^-) = 0 \quad (14)$$

$$\mathbf{u}(t) \in \mathcal{U} \subset \mathbb{R}^{n_u},\ \ \mathbf{w} \in \{0,1\}^{n_w},\ \ \mathbf{x}(t) \in \mathcal{X} \subset \mathbb{R}^{n_x}, \qquad (15)$$

$$0 \leq \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}, t), t \in [t_a, t_e] \quad \textit{inequality constraints}, \quad (16)$$

$$\mathbf{x}(t_a) = \mathbf{x}_a \quad \textit{initial conditions}, \qquad (17)$$

$$\mathbf{x}(t_e) = \mathbf{x}_e \quad \textit{terminal conditions}, \qquad (18)$$

where the initial time t_a and the final time t_e are free or fixed. The Mayer type part Θ of the performance index is a general function of the phase transition times (events) t_i, $i = 0, \dots, N$, of the continuous $\mathbf{x}(t_i^-)$, $\mathbf{x}(t_i^+)$ and of the binary $\mathbf{w}$

$$\Theta := \Theta[\ \mathbf{x}(t_0^-), \mathbf{x}(t_0^+), \dots, \mathbf{x}(t_N^-), \mathbf{x}(t_N^+); \mathbf{w}; t_0, \dots, t_N]\ .$$

Here, $t_a = t_0$, $t_e = t_N$ may be specified while the number of phases N may be given or free. The integrand ψ is a real-valued function of the continuous and binary state and control variables and of time.

A branch-and-bound technique is applied to efficiently search the entire solution space performing a truncated binary tree search for the binary variable:

- At the root of the binary search tree, all binary variables are relaxed, i.e., $\mathbf{w} \in [0,1]^{n_w}$.
- In each inner node some of the binary variables are 0 or 1, all others are relaxed.
- At the leaves of the search tree, all binary variables are 0 or 1.
- The partially relaxed binary variables at an inner node define a hybrid optimal control problem whose numerical solution by DIRCOL yields a lower bound on the hybrid performance index for all nodes at subsequent branches. An upper bound is obtained by solving a hybrid optimal control problem for a not relaxed binary variable.
- If the hybrid performance index associated with a lower bound in a node is greater than the current best upper bound of the whole search tree, than all subsequent branches from this node can be cut off.

This approach solves the hybrid optimal control problem to the global optimum assuming the resulting multi-phase control problems are solved optimally. However, the performance of the method is very sensitive to the initial guess for the binary variables. Thus, further research is needed for

- methods to find a good initial estimate of the discrete control variables, i.e., a good upper bound on the hybrid optimal control performance index, and for
- improved estimates of lower bounds on the hybrid optimal control performance index, e.g., by applying necessary or sufficient conditions derived from a hybrid HJCB partial differential equation or inequality, or from a hybrid Maximum Principle using the adjoint variable estimates provided by DIRCOL. However, both approaches yet need to be established through future investigation.

Another drawback of the proposed approach is that the existence of solutions to the relaxed problems must be ensured, where the relaxed problem may not be of any physical significance with respect to the underlying application.

4.4 Suboptimal hybrid solution approach

By choosing fixed values for the transition times and states it is possible to decompose the multi-phase optimal control problem into several TPBVPs, because the coupling via transversality conditions being part of the optimality conditions vanishes. Problematic in this approach is that the choice for a priori unknown transition times and states can be far away from the optimal choice when using heuristics.

By assuming fixed values of transition times and states on a grid, one can define a (large) set of TPBVPs with fixed initial and terminal conditions, which can then be solved numerically. Further, by assuming that one can compute these solutions very efficiently, it is possible to construct a weighted graph between all the grid nodes with the optimal cost on its vertices. Finally, the suboptimal solution to the original hybrid optimal control problem can be found in this weighted graph using a graph search algorithm.

The advantage of this method is that one often has an insight into what are reasonable transition times and states by knowledge about the physics of the application. A major disadvantage is that the number of TPBVPs increases with the number of grid nodes for transition times and states higher than at polynomial order. On the other hand it requires significantly less effort compared to a straightforward dynamic programming approach which additionally discretizes all states and times.

Many interesting theoretical questions for this approach remain unanswered, e.g., a quantitative measure (estimation) of the distance from optimality, if one should reduce to TPBVPs or treat some of the phases together in a MPBVP with free transition conditions.

The approach has been successfully used in determining suboptimal solutions in complex mechatronic situations. Even though the suboptimal solution may be far off the optimum, the method can at least provide practically useful suboptimal solutions, which may be a good initial starting point for other search procedures.

5 Application examples

5.1 Underactuated robot R2D1

In the following solutions to the hybrid optimal control problem of R2D1 previously defined in Example 1, we consider the case of the specific initial and terminal conditions

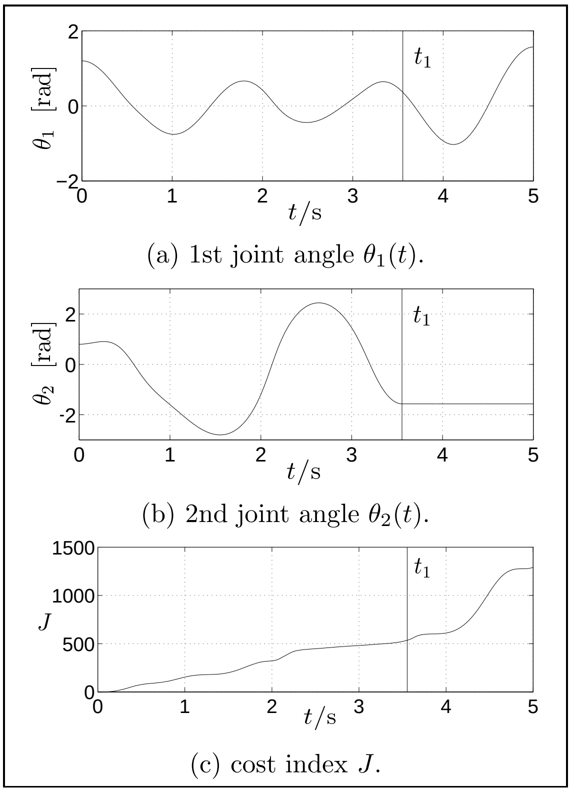
$$\mathbf{x}(0) = \begin{bmatrix} 1.2 \\ 0 \\ 0.8 \\ 0 \end{bmatrix}, \ \mathbf{x}(t_e) = \begin{bmatrix} \pi/2 \\ 0 \\ -\pi/2 \\ 0 \end{bmatrix},$$

with the discrete state of the holding brake as $q(0) = 1$, $q(t_e) = 2$. The terminal time is fixed $t_e = 5$ and $\alpha = 20$.

Optimal solutions for two and four phases were computed using DIRCOL discussed in Section 4.2. The joint velocity $x_4 = \dot{\theta}_2 = 0$ is constrained to zero at the internal switching points when the holding brake is activated once/twice in the 2/4-phase problem.

Figures 6, 7 show the optimal trajectories for the two and four phase problems, respectively. The switching points are indicated by vertical lines in the plots.

In Figure 6, the swinging motion used to increase the velocity $\dot{\theta}_2$ until time t_1 can be observed. The desired zero velocity condition $\dot{\theta}_2(t_1) = 0$ holds at the time t_1

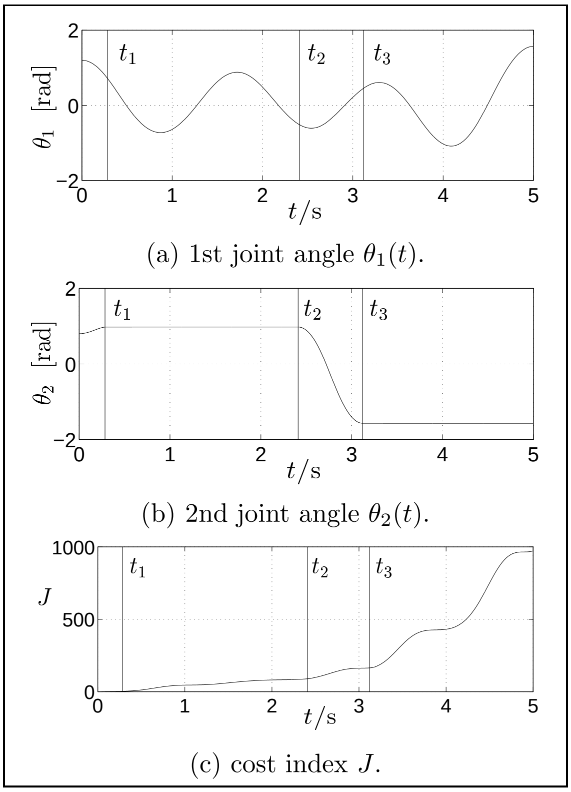


(a) 1st joint angle $\theta_1(t)$.

(b) 2nd joint angle $\theta_2(t)$.

(c) cost index J.

Figure 6: Optimal trajectories of the two phase solution.



(a) 1st joint angle $\theta_1(t)$.

(b) 2nd joint angle $\theta_2(t)$.

(c) cost index J.

Figure 7: Optimal trajectories of the four phase solution.

when the holding brake is closed. Note that the desired angle of $\theta_2 = -\pi/2$ is achieved at the time of switching; the second joint does not move in the second phase. The desired final state is reached at $t_e = 5$. Figure 6(c) shows the cost index versus time. The optimal cost index value is 1292.

When solving the four phase problem the holding brake is switched off⇝on⇝off⇝on. Figure 7 shows the optimal solution. When comparing the results, it is clear that the trajectories of the four phase problem are "smoother" and better "controlled", cf. Figures 6(b) and 7(b). The first joint again shows a swinging motion. A significant improvement of the four phase problem is a 25% reduction in overall cost which is now only 972.

This result clearly shows that an important question of hybrid optimal control problems is to establish the optimal number of phases.

5.2 Motorized traveling salesman

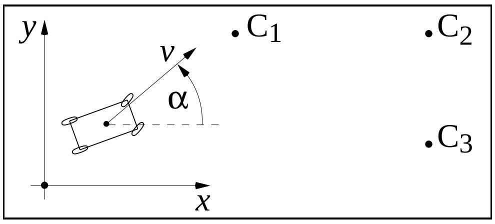


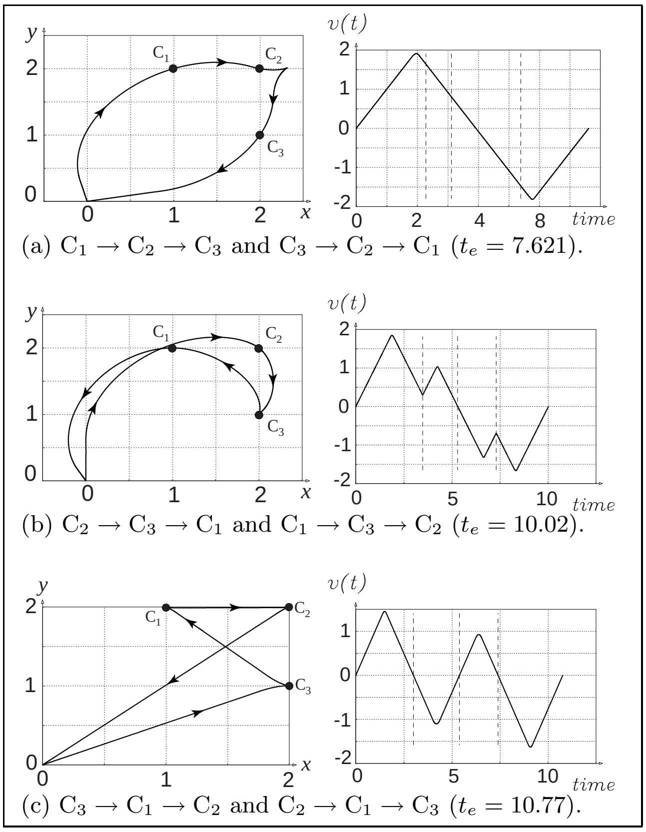
Figure 8: The motorized traveling salesman.

Example 3 *A salesman spends his time visiting n_c cities cyclically. In one tour he visits each city just once and finishes up in the origin where he started. In what order should he visit them to minimize the overall travel time?* The Traveling Salesman Problem (TSP) is one of the most prominent members of combinatorial optimization problems [18]. Here, we investigate a dynamical extension of the TSP as suggested in [32] to demonstrate the strong interaction of continuous and discrete dynamics in hybrid optimal control. The salesman is supposed to drive a car (Figure 8). The task is to determine the steering angle velocity γ and the accelerating or braking force β (continuous controls) and the order (discrete control) in which the n_c cities $\mathrm{C}_k = (x_k^c, y_k^c)^T$, $k = 1, \ldots, n_c$, have to be visited such that the overall travel time is minimized. There are no further restrictions on the path, i.e., the "road", in the (x, y)-plane.

A simplified kinematical model of the car is given by

$$\begin{aligned} \dot{x}(t) &= v(t)\cos(\alpha(t)), & x(0) &= 0 = x(t_e), \\ \dot{y}(t) &= v(t)\sin(\alpha(t)), & y(0) &= 0 = y(t_e), \\ \dot{v}(t) &= \beta(t), & v(0) &= 0 = v(t_e), \\ \dot{\alpha}(t) &= \gamma(t), & \alpha(0),\ &\alpha(t_e) \ \text{ free.} \end{aligned} \tag{19}$$

A phase describes the travel between two cities. Thus, the number N of phases is equal to $n_c + 1$. Let t_i^c denote the time when the i-th city is passed. Then the motorized TSP problem is formulated as a MIOCP according



(a) $C_1 \to C_2 \to C_3$ and $C_3 \to C_2 \to C_1$ ($t_e = 7.621$).

(b) $C_2 \to C_3 \to C_1$ and $C_1 \to C_3 \to C_2$ ($t_e = 10.02$).

(c) $C_3 \to C_1 \to C_2$ and $C_2 \to C_1 \to C_3$ ($t_e = 10.77$).

Figure 9: Minimum time tour candidates for 3 cities (left: tour in the (x, y)-plane, right: velocity v); arrows indicate forward tour; minimum time problem solution is $C_1 \to C_2 \to C_3$ (or backwards $C_3 \to C_2 \to C_1$) in (a).

to Definition 2 by $\mathbf{u} = (\gamma, \beta)^T$, $\mathbf{x} = (x, y, v, \alpha)^T$ and

$$\min_{\mathbf{u},\,\mathbf{w}} J(\mathbf{u}, \mathbf{w}) := t_e \tag{20}$$

$$s_j(\mathbf{x}, \mathbf{u}, \mathbf{w}, t_i^c - 0) := \begin{pmatrix} x(t_i^c - 0) \\ y(t_i^c - 0) \end{pmatrix} - \sum_{k=1}^{n_c} w_{i,k} \begin{pmatrix} x_k^c \\ y_k^c \end{pmatrix} \tag{21}$$

$$\mathbf{x}(t_i^c + 0) = \phi_j(\mathbf{x}, \mathbf{u}, \mathbf{w}, t_i^c - 0) := \mathbf{x}(t_i^c - 0) \tag{22}$$

$$\sum_{i=1}^{n_c} w_{i,k} = 1, \quad \sum_{k=1}^{n_c} w_{i,k} = 1, \quad 0 \le w_{i,k} \le 1. \tag{23}$$

The latter constraints ensure that each city is visited exactly once on each tour.

Each tour is a permutation of n_c cities. Also the problem is autonomous. Thus, a tour driven forwards or backwards yields the same travel time. Therefore, the number of possible tours is $(n_c)!/2$. The number of tours increases not polynomially with the number of cities. For example, for 3 cities the number of tours is 3, for 5 cities it is 60, for 10 cities it is 1 814 400, and for 50 cities it is approximately $\approx 1.52 \times 10^{64}$. Now if we assume that all tours for 5 cities can be computed in one second inclusive the selection of the best one, then to solve the problem for 20 cities in this way will need approximately $20!/5! \approx 2.03 \times 10^{16}\,\mathrm{s} \approx 643$ million years. The TSP is NP-complete!

For $n_c = 3$ cities $C_1 = (1, 2)$, $C_2 = (2, 2)$, $C_3 = (2, 1)$, the three possible tours are displayed in Figure 9. For each of the tours the continuous controls have been optimized using DIRCOL with respect to the terminal time for a given discrete variable, i.e., order of cities, i.e., sequence of phases. The binary search for the optimal binary and continuous controls is quickly finished for only 3 cities. The general branch-and-bound tree search combined with the multi-phase optimal control solution by DIRCOL described in Section 4.3 is currently being implemented.

6 Conclusions

After defining a rather general class of hybrid optimal control problems, we presented a typical example of a two-phase mechatronic application to illustrate the novel challenges. The two main solution approaches to conventional optimal control problems — (i) Hamilton-Jacobi-Carathéodory-Bellman PDEs and dynamic programming, (ii) Euler-Lagrange differential equations and the Maximum Principle — were briefly summarized. Extensions of these approaches to multi-phase optimal control problems were discussed for the case when the optimal discrete state sequence is a priori known or given as part of the problem. The key challenge when solving hybrid optimal control problems is that neither the order of the phases, nor the times of phase transitions, nor their number are known and therefore have to be determined when solving the problem. A hybrid version of the method of dynamic programming based on a hybrid formulation of the principle of optimality was illustrated and the difficulties arising from its application to solving hybrid optimal control problems were presented.

Another central part of the article discussed numerical issues when solving hybrid optimal control problems. Again, a brief historic review of direct and indirect methods is followed by the description of more recent work, such as the direct collocation method DIRCOL to solve multi-phase problems. With this approach, optimal control problems are transcribed to nonlinear parameter optimization problems subject to collocation constraints (point-wise fulfillment of the differential and constraint equations) resulting in a set of equality and inequality constraints for the nonlinear program. DIRCOL in its most recent version is extremely efficient by exploiting Jacobian sparsity of the resulting parametric optimization problem; this is accomplished by relying on the advanced sparse SQP solver SNOPT. The numerical robustness of DIRCOL is another important aspect.

On the basis of DIRCOL, a new paradigm to solving hybrid optimal control problems has become feasible, i.e. an inner/outer iteration scheme. In the inner iteration DIRCOL is used to solve a specific multi-phase problem for a fixed discrete state sequence of the original hybrid problem. The resulting optimal cost index is then used in an outer iteration to decide which discrete state sequence to solve next. Within this class of inner/outer iterative algorithms, the two approaches of i) branch-and-bound combined with (binary) discrete variable relaxation, and ii) suboptimal solution by choice of fixed transition times and states (possibly on a grid) at phase transitions, have been presented. Both approaches have in common that a significant number of outer iterations is not regarded as problematic because of the efficiency with which the solutions to the multi-phase problems of the inner iteration are obtainable by DIRCOL.

Using the proposed numerical methods, two application examples were solved. For the underactuated robot R2D1, optimal 2/4-phase solutions were directly computed by DIRCOL. Remarkable is that in this case the 4-phase solution resulted in a cost reduction of approximately 25% which illustrates the importance of determining the optimal number of phases when solving hybrid optimal control problems. The purely combinatorial TSP was extended to the hybrid dynamic case with the motorized traveling salesman problem. For a very limited number of cities together with their connected number of multi-phase problems, we presented the optimal solutions obtained by DIRCOL.

Fact is — despite all the overview and discussion in this article as well as consideration of the most recent publications in this area — that hybrid optimal control is a widely open field of research with many of the main theoretical and practical questions being still unanswered. Some preliminary numerical methods were suggested here; however, many theoretical issues connected to these remain. The methods need to be applied to more problems from various application domains to inspire further research and development of this exciting field of hybrid optimal control.

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