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Active Suspension Design For A Tractor By Optimal Control Methods

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Summary

An active suspension system for improving the ride comfort and safety of a tractor is investigated. The underlying planar dynamic tractor model and a suitable objective for optimal suspension are introduced. The problem of optimal active suspension leads to a linear-quadratic optimal control problem. Classical Linear-Quadratic Regulator (LQR) theory provides a closed-loop control for the steady state problem which is optimal only for an initial disturbance input from the road. A direct transcription method can handle more general disturbances and models but provides only an open-loop solution, where the time history of the optimal control is given along the optimal trajectory for one type of deterministic disturbance and initial value only. Simulation results for two different road disturbances are given comparing both approaches.

Key Words: optimal active suspension; steady-state LQR problem; closed-loop solution; open-loop solution; direct transcription method

1 Introduction

Back problems as spinal affection are a common disease for agriculturists driving tractors. One reason is that the rear axle of a tractor usually has

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no suspension dampers, which seems to be contrary to the otherwise highly technical standard of modern tractors. The only vibration damping of the rear axle is by the deflection of the large tires of the rear wheels. However, with a perfectly adjusted front axle suspension, ride comfort and ride safety can be improved significantly.[15] To improve the ride comfort further and the overall stiffness to resist body forces – especially in the case of tractors with a heavy rear-mounted implement (a load or a work tool) – an active suspension system for the front axle could be used in principle in combination with an active control of the rotational motion of the implement (Figure 1).

Actively controlled suspension systems have gained increasing interest in automotive engineering research during the last two decades. The superiority of active suspension systems to passive suspension systems has been shown.[9,21,22,24] In the case of a passive suspension system, the vibrational behavior of a vehicle for various excitations from road disturbances is given by the stiffness and the damping rates of the shock absorbers. In the case of an active suspension system, the stiffness and damping properties of the shock absorber can be controlled, e. g., by a hydraulic actuator. When *optimal* active suspension systems are addressed in literature, most often this problem is formulated as a linear-quadratic optimal control problem using a quarter car model consisting of two bodies, namely the chassis and a wheel. Regarding the steady-state LQR problem, the well-known Riccati-equation solution provides an optimal feedback-control law.[8]

But, as it is shown later on, the steady-state solution is optimal for a limited class of road disturbances only. Optimal open-loop controls for arbitrary disturbances as well as for more general optimization criterions and subject to nonlinear differential equations and general constraints can be computed with a direct transcription method. The direct collocation method DIRCOL[18,19] will be introduced and numerical solutions are compared to the LQR solution for a given initial value problem and different disturbances. For simulation, a planar tractor model[16] is used.

2 Dynamic tractor model and objective for suspension

In the sequel, a planar model of a tractor with a suspended front axle and a rotable rear-mounted implement[16] will be investigated in detail. The mechanical system consists of three bodies, namely vehicle body, front axle, and rear-mounted implement. It exhibits four degrees of freedom, which are the generalized coordinates of the system: vertical displacement z_V[m] of the

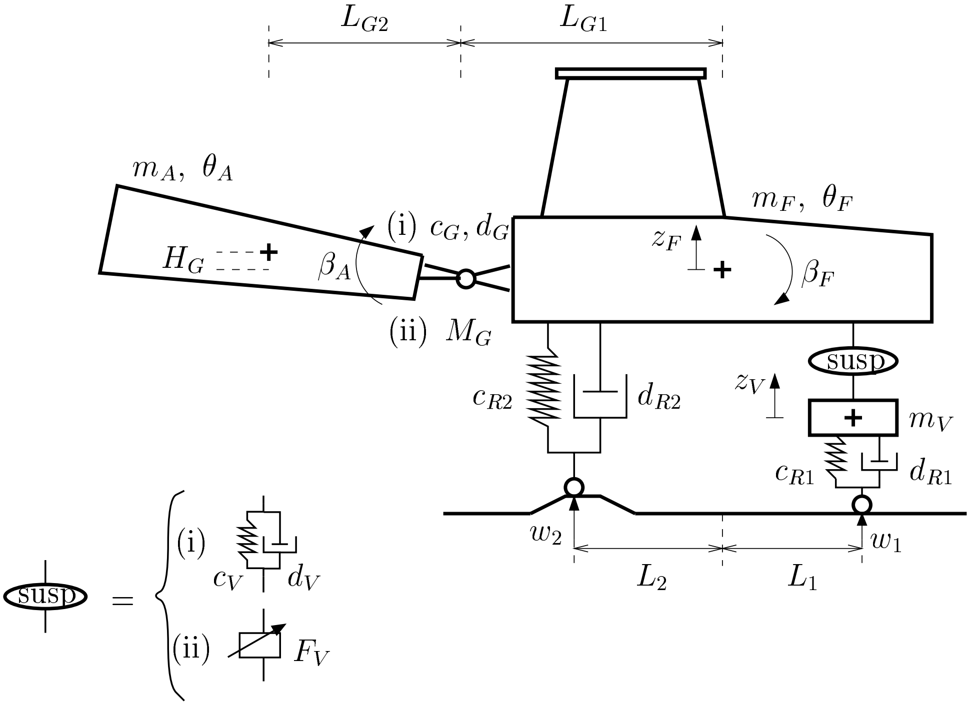

Figure 1. Planar tractor model with (i) passive suspension,
(ii) active suspension.

front axle, vertical displacement z_F[m] of the vehicle body, rotational motion β_F[rad] of the vehicle body and rotational motion β_A[rad] of the implement relative to the vehicle body. The dynamical behavior of this multi-body-system is described by the equations of motion

$$\mathbf{M}\,\ddot{\mathbf{z}}(t) = \mathbf{q}(t, \mathbf{z}, \dot{\mathbf{z}}) \tag{1}$$

with $\mathbf{z}^T = (z_V, z_F, \beta_F, \beta_A) \in \mathrm{I\!R}^4$. The mass matrix $\mathbf{M} \in \mathrm{I\!R}^{4\times4}$ is constant, symmetric and has the lower triangle

$$\mathbf{M}_{low} = \begin{pmatrix} m_V & & & \\ 0 & m_A + m_F & & \\ 0 & m_A\,L_G & \theta_A + \theta_F + m_A(L_G^2 + H_G^2) & \\ 0 & m_A\,L_{G2} & \theta_A + m_A\,L_G\,L_{G2} & \theta_A + m_A\,{L_{G2}}^2 \end{pmatrix}$$

where $L_G := L_{G1} + L_{G2}$. The excitation $\mathbf{q}(t, \mathbf{z}, \dot{\mathbf{z}}) \in \mathbb{R}^4$ is given by

$$\mathbf{q} = \begin{pmatrix} P_1 & - & F_V & & - & m_V\, g \\ P_2 & + & F_V & & - & (m_F + m_A)\, g \\ L_2\, P_2 & - & L_1\, F_V & & - & m_A\, L_G\, g \\ & & & M_G & - & m_A\, L_{G2}\, g \end{pmatrix} . \tag{2}$$

The values of the constants are listed in Table 1.

Table 1: Parameters of the tractor model.

Quantity		Parameter	Value	Unit
vertical distances:				
body (center of mass)	– front wheel	L_1	1.400	m
body	– rear wheel	L_2	1.450	m
body	– rotational joint	L_{G1}	2.000	m
implement (center of mass)	– rotational joint	L_{G2}	1.500	m
horizontal distance:				
body	– implement	H_G	0.100	m
mass of the vehicle body		m_F	$9.0 \cdot 10^3$	kg
mass of the implement		m_A	$1.0 \cdot 10^3$	kg
mass of the front axle		m_V	$5.0 \cdot 10^2$	kg
inertia of the vehicle body		θ_F	$6.0 \cdot 10^4$	Nms^2
inertia of the implement		θ_A	$5.0 \cdot 10^3$	Nms^2
stiffness of the front wheel		c_{R1}	$1.0 \cdot 10^6$	N/m
stiffness of the rear wheel		c_{R2}	$1.5 \cdot 10^6$	N/m
stiffness of the front axle		c_V	$1.0 \cdot 10^4$	N/m
stiffness of the rotational joint		c_G	$6.4 \cdot 10^5$	N/m
damping rate of the front wheel		d_{R1}	$7.0 \cdot 10^3$	Ns/m
damping rate of the rear wheel		d_{R2}	$9.0 \cdot 10^3$	Ns/m
damping rate of the front axle		d_V	$1.0 \cdot 10^5$	Ns/m
damping rate of the rotational joint		d_G	$2.8 \cdot 10^4$	Ns/m
acceleration by gravity		g	9.81	m/s^2

Defining the state vector

$$\mathbf{x}^T := (\mathbf{z}^T, \dot{\mathbf{z}}^T)\,, \tag{3}$$

the front wheel load force P_1[N] and the rear wheel load force P_2[N] depend linearly on the states by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \mathbf{C}\,(\mathbf{x} - \mathbf{d})$$

with

$$\mathbf{C} = \begin{pmatrix} -c_{R1} & 0 & 0 & 0 & -d_{R1} & 0 & 0 & 0 \\ 0 & -c_{R2} & -c_{R2}L_2 & 0 & 0 & -d_{R2} & -d_{R2}L_2 & 0 \end{pmatrix}.$$

The road disturbance is $\mathbf{d}^T(t) = (w_1(t), w_2(t), 0, 0, \dot{w}_1(t), \dot{w}_2(t), 0, 0) \in \mathbb{R}^8$.

In the case of a passive suspension system, the state-dependent functions of the front axle suspension force F_V[N] and the moment of the rotational joint M_G[Nm] are determined by the stiffness and damping parameters c_V, c_G, d_V and d_G:

$$\begin{pmatrix} F_V \\ M_G \end{pmatrix} = \begin{pmatrix} c_V & -c_V & c_V L_1 & 0 & d_V & -d_V & d_V L_1 & 0 \\ 0 & 0 & 0 & -c_G & 0 & 0 & 0 & -d_G \end{pmatrix} \mathbf{x}\,. \tag{4}$$

By optimization of the parameters c_V, c_G, d_V, d_G within certain bounds an optimal passive suspension system can be obtained for the ride over a given road disturbance.[16] In order to improve the ride comfort and the overall stiffness to resist body forces the objective of the optimization in Reference 16 is to minimize the relative steady-state errors of the wheel load forces and the relative acceleration of the vehicle body. The steady-state values of the forces P_1, P_2, F_V and the moment M_G, which are the solution of the system of equations given by (2) if $\mathbf{q} = \mathbf{0}$, are approximately

$$\begin{aligned} P_{1\,stat} &= 42768.158 \text{ N}\,, & P_{2\,stat} &= 60236.842 \text{ N}\,, \\ F_{V\,stat} &= 37863.158 \text{ N}\,, & M_{G\,stat} &= 14715.0 \quad \text{Nm}\,. \end{aligned}$$

Then (2) is equivalent to

$$\mathbf{q} = \begin{pmatrix} P_1 - P_{1\,stat} & - & (F_V - F_{V\,stat}) \\ P_2 - P_{2\,stat} & + & F_V - F_{V\,stat} \\ L_2(P_2 - P_{2\,stat}) & - & L_1(F_V - F_{V\,stat}) \\ & & M_G - M_{G\,stat} \end{pmatrix}$$

and the optimization criterion for the passive suspension design reads as

$$J_{passive} = \int_{t_0}^{t_f} \left(\left(\frac{P_{1\,dyn}}{P_{1\,stat}}\right)^2 + \left(\frac{P_{2\,dyn}}{P_{2\,stat}}\right)^2 + \left(\frac{\ddot{z}_F}{g}\right)^2 \right) \mathrm{d}t \tag{5}$$

with the increments of the dynamic wheel load forces $P_{1\,dyn} := P_1 - P_{1\,stat}$ and $P_{2\,dyn} := P_2 - P_{2\,stat}$.

In the case of an active suspension system, the time dependent controls F_V and M_G have to be determined in a proper way, e. g., by minimization of a suitable objective. Here, besides safety and comfort the objective has to take into account another, contrary goal, namely to minimize the control effort required. Therefore, the incremental controls $\mathbf{u}^T = (F_{V\,dyn}, M_{G\,dyn}) := (F_V - F_{V\,stat}, M_G - M_{G\,stat})$, which are the deviations from the steady-state controls, are included in the optimization criterion after division by their steady-state values:

$$\tilde{J}_{active} = \int_{t_0}^{t_f} \left(\left(\frac{P_{1\,dyn}}{P_{1\,stat}} \right)^2 + \left(\frac{P_{2\,dyn}}{P_{2\,stat}} \right)^2 + \left(\frac{\ddot{z}_F}{g} \right)^2 + \left(\frac{F_{V\,dyn}}{F_{V\,stat}} \right)^2 + \left(\frac{M_{G\,dyn}}{M_{G\,stat}} \right)^2 \right) \mathrm{d}t \,.$$

However, for $\tilde{J}_{active}$ as objective of an optimal active suspension system no stable solution exists (see Remark 1 of Section 5). Three of the four degrees of freedom $z_V, z_F, \beta_F, \beta_A$ do not reach their steady-state value zero. Therefore, $\tilde{J}_{active}$ is augmented by the rotational motion of the implement relative to the vehicle body β_A and the vertical displacement of the spring of the front axle relative to the vehicle body

$$z_{V rel} = -z_V + z_F - L_1 \beta_F = (\,-1 \quad 1 \;\; -L_1 \quad 0)\;\mathbf{z} \,.$$

Both are incremental quantities with steady-state values of zero. For weighting their values in the performance index, they are divided by suitable choosen constants $\beta_{A\max}$ and $z_{V\max}$, respectively. The values

$$z_{V\max} = 0.025 \,, \quad \beta_{A\max} = 0.105$$

have been used.[16] Then, the optimization criterion finally results in

$$J_{active} = \tilde{J}_{active} + \int_{t_0}^{t_f} \left(\left(\frac{z_{V rel}}{z_{V\max}} \right)^2 + \left(\frac{\beta_A}{\beta_{A\max}} \right)^2 \right) \mathrm{d}t \,. \qquad (6)$$

3 Linear-quadratic optimal control problem

The described problem for the optimal control $\mathbf{u}$ of an active suspension system can be formulated as a linear-quadratic optimal control problem.

The basic control problem consists of a linear system of first order differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{G}\,\mathbf{d}(t) \qquad (7)$$

and a quadratic performance index

$$\begin{aligned} J[\mathbf{u}, t_f] &= \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)\ \mathrm{d}t = \\ &= \int_{t_0}^{t_f} \Big(\mathbf{x}^T(t)\,\mathbf{Q}\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathbf{R}\,\mathbf{u}(t) + 2\,\mathbf{x}^T(t)\,\mathbf{S}\,\mathbf{u}(t)\Big)\ \mathrm{d}t\,. \qquad (8) \end{aligned}$$

The matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{G} \in \mathbb{R}^{n\times p}$, $\mathbf{Q} \in \mathbb{R}^{n\times n}$, $\mathbf{R} \in \mathbb{R}^{m\times m}$ and $\mathbf{S} \in \mathbb{R}^{n\times m}$ are all time-invariant. $\mathbf{x}(t) \in \mathbb{R}^n$ describes the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ the control vector and $\mathbf{d}(t) \in \mathbb{R}^p$ the vector of inputs from the road disturbance.

Now we derive the optimal control problem corresponding to the planar tractor model. The dimensions are $n = 8, m = 2$ and $p = 8$. Using definition (3) and the regularity of the mass matrix $\mathbf{M}$ the second order system (1) is transformed into the first order system

$$\dot{\mathbf{x}} = \begin{pmatrix} (\mathbf{O}_{(4,4)} \quad \mathbf{I}_{(4,4)})\ \mathbf{x} \\ \mathbf{M}^{-1}\ \mathbf{q} \end{pmatrix} \qquad (9)$$

with zero matrix $\mathbf{O}_{(4,4)} \in \mathbb{R}^{4\times 4}$ and identity matrix $\mathbf{I}_{(4,4)} \in \mathbb{R}^{4\times 4}$. Since $\mathbf{q}$ depends linearly on $\mathbf{x}$ and $\mathbf{u}$,

$$\mathbf{q} = (\mathbf{C_R} \quad \mathbf{D_R})\,(\mathbf{x} - \mathbf{d}) + \tilde{\mathbf{B}}\,\mathbf{u}$$

with

$$\mathbf{C_R} = \begin{pmatrix} -c_{R1} & 0 & 0 & 0 \\ 0 & -c_{R2} & -c_{R2}\,L_2 & 0 \\ 0 & -c_{R2}\,L_2 & -c_{R2}\,{L_2}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D_R} = \begin{pmatrix} -d_{R1} & 0 & 0 & 0 \\ 0 & -d_{R2} & -d_{R2}\,L_2 & 0 \\ 0 & -d_{R2}\,L_2 & -d_{R2}\,{L_2}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{B}} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ -L_1 & 0 \\ 0 & 1 \end{pmatrix},$$

the dynamic system (9) is linear and can be written in the form of Equation (7). The system matrices $\mathbf{A}, \mathbf{B}, \mathbf{G}$ are given by

$$\mathbf{A} = \begin{pmatrix} \mathbf{O}_{(4,4)} & \mathbf{I}_{(4,4)} \\ \mathbf{M}^{-1}\mathbf{C_R} & \mathbf{M}^{-1}\mathbf{D_R} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \mathbf{O}_{(4,2)} \\ \mathbf{M}^{-1}\tilde{\mathbf{B}} \end{pmatrix}, \mathbf{G} = \begin{pmatrix} \mathbf{O}_{(4,4)} & \mathbf{O}_{(4,4)} \\ -\mathbf{M}^{-1}\mathbf{C_R} & -\mathbf{M}^{-1}\mathbf{D_R} \end{pmatrix}.$$

The performance index of Equation (6) is equivalent to the form of Equation (8) using the matrices

$$\begin{aligned}
\tilde{\mathbf{Q}} &= \mathbf{C}^T \mathrm{diag}\left(\frac{1}{P_{1\,stat}^2}, \frac{1}{P_{2\,stat}^2}\right) \mathbf{C} + \frac{1}{g^2}\mathbf{A}^T \mathbf{e_6} \mathbf{e_6}^T \mathbf{A}\,, \\
\mathbf{Q} &= \tilde{\mathbf{Q}} + \begin{pmatrix} \tilde{\mathbf{B}}\ \mathrm{diag}\left(\frac{1}{z_{V\max}^2}, \frac{1}{\beta_{A\max}^2}\right) \tilde{\mathbf{B}}^T & \mathbf{O}_{(4,4)} \\ \mathbf{O}_{(4,4)} & \mathbf{O}_{(4,4)} \end{pmatrix}, \\
\mathbf{R} &= \frac{1}{g^2}\mathbf{B}^T \mathbf{e_6} \mathbf{e_6}^T \mathbf{B} + \mathrm{diag}\left(\frac{1}{F_{V\,stat}^2}, \frac{1}{M_{G\,stat}^2}\right), \\
\mathbf{S} &= \frac{1}{g^2}\mathbf{A}^T \mathbf{e_6} \mathbf{e_6}^T \mathbf{B}\,,
\end{aligned}$$

and defining $\mathbf{e_6}^T := (0,0,0,0,0,1,0,0) \in \mathbb{R}^8$.

4 Solution of the optimal control problem

4.1 Riccati solution of the steady-state regulator problem

An optimal control problem given in the form of Eqs. (7) and (8) is called a disturbance-rejection problem.[8] The objective is to determine the control input that minimizes the effect of the additive disturbance signal $\mathbf{G}\,\mathbf{d}(t)$ on the value of the performance index.

If $\mathbf{G}\,\mathbf{d}(t) = \mathbf{0}$ for $t > t_0$, the objective is to maintain the state vector $\mathbf{x}$ close to zero. Since G has non-zero elements, this is the case for a step input at initial time only, which can be transformed into an initial condition

$$\begin{aligned} \mathbf{d}(t_0) &= \mathbf{d_0}\,, \\ \mathbf{d}(t) &= \mathbf{0}\,, \quad t_0 < t \le t_f\,. \end{aligned} \tag{10}$$

Then, the above problem is referred to as optimal linear-quadratic regulator (LQR) problem. More specifically, since the matrices $\mathbf{A}, \mathbf{B}, \mathbf{Q}$ and $\mathbf{R}$ are time-invariant it is a so-called steady-state LQR problem assuming t_f approaches infinity. Investigation of the matrices $\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ for the tractor model shows that

- $\mathbf{Q} = \mathbf{D}^T\mathbf{D}$,
- $\mathbf{R}$ is positive definite and symmetric,
- $\mathbf{Q} - \mathbf{S}\,\mathbf{R}^{-1}\mathbf{S}^T$ is non-negative definite and symmetric,
- $(\mathbf{A}, \mathbf{B})$ is controllable and
- $(\mathbf{A}, \mathbf{D})$ is observable.

Then, a unique solution of the steady-state LQR problem exists, and the optimal closed-loop system $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\,\mathbf{K})\mathbf{x}$ is asymptotically stable. [8] The optimal feedback-control law reads as

$$\mathbf{u} \;=\; -\,\mathbf{K}\,\mathbf{x} \;=\; -\,\mathbf{R}^{-1}(\mathbf{B}^T\,\mathbf{P} + \mathbf{S}^T)\,\mathbf{x}\,, \tag{11}$$

where $\mathbf{P}$ is the unique positive definite symmetric solution of the algebraic matrix Riccati equation

$$\begin{array}{rlcll} & \mathbf{A}^T\,\mathbf{P} + \mathbf{P}\,\mathbf{A} + \mathbf{Q} - (\mathbf{P}\,\mathbf{B} + \mathbf{S})\,\mathbf{R}^{-1}(\mathbf{P}\,\mathbf{B} + \mathbf{S})^T & & & = \;0 \\ \Longleftrightarrow & (\mathbf{A} - \mathbf{B}\,\mathbf{R}^{-1}\mathbf{S}^T)^T\,\mathbf{P} & + & \mathbf{P}\,(\mathbf{A} - \mathbf{B}\,\mathbf{R}^{-1}\mathbf{S}^T) \; + & \\ & (\mathbf{Q} - \mathbf{S}\,\mathbf{R}^{-1}\mathbf{S}^T) & - & \mathbf{P}\,\mathbf{B}\,\mathbf{R}^{-1}\mathbf{B}^T\,\mathbf{P} & = \;0\,. \end{array} \tag{12}$$

The numerical solution of the Riccati equation results in the rounded feedback matrix

$$\mathbf{K} \;=\; 10^4\cdot$$
$$\begin{pmatrix} -101.633 & 132.748 & -206.156 & -1.658 & -2.125 & 9.988 & -21.924 & -1.167 \\ -0.656 & 0.609 & -9.196 & 13.984 & -0.005 & 0.367 & 2.774 & 4.536 \end{pmatrix}.$$

On the other hand, the minimum principle from optimal control theory [13] results in

$$\mathbf{u} \;=\; -\,\frac{1}{2}\,\mathbf{R}^{-1}\,(\mathbf{B}^T\,\lambda + 2\,\mathbf{S}^T\,\mathbf{x})$$

with the vector of adjoint or co-state variables $\lambda^T := \frac{\partial \mathbf{J}}{\partial \mathbf{x}}$. Comparison to Equation (11) shows that the solution $\mathbf{P}$ of the matrix Riccati equation also provides an expression for the vector λ of the adjoint variables according to

$$\lambda(t) \;=\; 2\,\mathbf{P}\,\mathbf{x}(t)\,. \tag{13}$$

4.2 Direct collocation method DIRCOL

DIRCOL [19] is a special direct transcription method. [14,20] By a discretization of state and control variables using piecewise polynomial approximations, the infinite dimensional optimal control problem in first order standard form

$$
\begin{array}{lll}
\text{minimize} & J[\mathbf{u}, t_f] = \phi(\mathbf{x}(t_f), t_f), \quad \phi : \mathbb{R}^{n+1} \to \mathbb{R} & \\
\text{subject to} & \text{the equations of motion} & \\
& \dot{x}_i(t) = f_i(\mathbf{x}(t), \mathbf{u}(t), t),\ i = 1, \ldots, n, \qquad t_0 \le t \le t_f, & \\
& \text{the boundary conditions} & (14) \\
& r(\mathbf{x}(0), \mathbf{x}(t_f), t_f) = 0, & \\
& \text{and the inequality constraints} & \\
& 0 \le g_j(\mathbf{x}(t), \mathbf{u}(t), t),\ j = 1, \ldots, m_g, \qquad t_0 \le t \le t_f. &
\end{array}
$$

is transcribed into a finite dimensional nonlinearly constrained optimization problem (NLP) for the parameters y of the discretizations of $\mathbf{x}$ and $\mathbf{u}$

$$
\begin{array}{ll}
\text{minimize} & \phi(y), \quad \phi : \mathbb{R}^{n_y} \to \mathbb{R} \\
\text{subject to} & a_i(y) = 0,\ i = 1, \ldots, m_e, \\
& b_i(y) \ge 0,\ i = 1, \ldots, m_i.
\end{array} \tag{15}
$$

The dimensions n_y, m_e, m_i mainly depend on the dimension n_d of the discretization grid $\Delta = (t_k)_{k=1}^{n_d}$

$$
t_0 = t_1 < t_2 < \ldots < t_{n_d - 1} < t_{n_d} = t_f. \tag{16}
$$

Here, the state variables are approximated by piecewise cubic polynomials x_{app} and are choosen to be continuously differentiable at the grid points. The control variables are approximated continuously by piecewise linear functions u_{app}. The differential equations have to be satisfied at the grid points t_k, t_{k+1} and at the center $t_{k+1/2}$ of each discretization grid (collocation at Lobatto points). [11]

The equality constraints $a(y)$ of the NLP result from the collocation conditions and the boundary conditions of the optimal control problem. The inequality constraints $b(y)$ result from the inequality constraints of the optimal control problem which have to be satisfied at the grid points of the discretization.

After a first approximation of the solution has been obtained for a first discretization grid, usually a sequence of refinement steps is applied in order to reduce local error estimates. Therefore, a sequence of related NLPs with

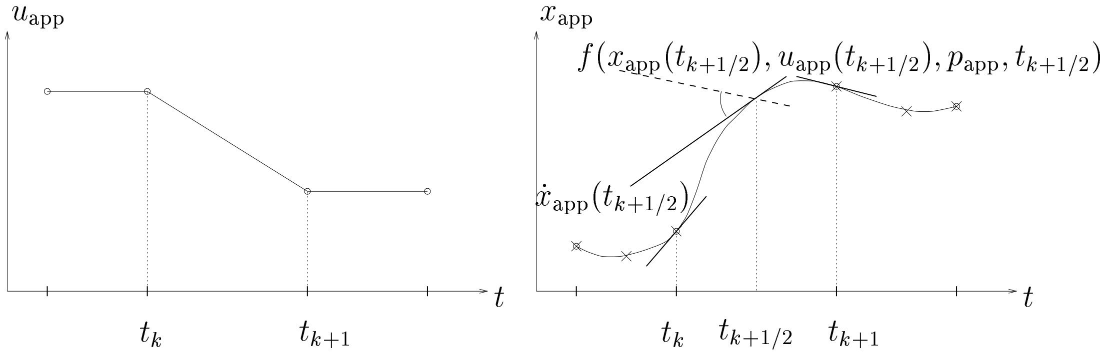

Figure 2. Discretization of control and state variables by piecewise polynomial functions.

usually increased dimensions has to be solved.[18] Each NLP is solved by the Sequential Quadratic Programming method NPSOL.[10]

Problems with a Lagrange-type objective, as in the problem of active suspension for a tractor,

$$J[\mathbf{u}, t_f] \;=\; \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)\ \mathrm{d}t \tag{17}$$

can be transformed into Mayer-type problems by introducing an additional state variable

$$\dot{x}_{n+1}(t) \;=\; L(\mathbf{x}(t), \mathbf{u}(t), t), \quad x_{n+1}(t_0) \;=\; 0, \quad x_{n+1}(t_f) \text{ free}, \tag{18}$$

in order to obtain the "new" objective

$$J[\mathbf{u}, t_f] \;=\; x_{n+1}(t_f) \;=\; \phi(\mathbf{x}^\star(t_f), t_f) \tag{19}$$

with the "new" state variable $\mathbf{x}^\star = (x_1, \ldots, x_n, x_{n+1})^T$ which is of dimension $n^\star = n + 1$.

The direct transcription method DIRCOL has been applied successfully for solving trajectory optimization problems from aeronautics, robotics and other fields.[18] Knowledge of optimal control theory or dealing with adjoint or co-state differential equations is not required by the user.[20] The user doesn't have to provide gradients of the model functions as they are approximated by finite differences. On the other hand, with a computed solution of the parameterized optimal control problem, a piecewise linear approximation of the histories of the adjoint variables and multipliers of constraints can be computed as well from the Lagrange multipliers of the NLP.[18]

5 Numerical Results

In this section, simulation results are given for the passive suspension design [16], cf. Equation (4), and the introduced active suspensions. Two types of road disturbances are considered: a step at initial time $t_0 = 0$ and a ramp. In the case of active suspension, the (closed-loop) steady-state LQR solution is compared to the (open-loop) solution provided by the direct collocation method DIRCOL.

The solution $\mathbf{P}$ of the algebraic matrix Riccati equation (12) and the matrix $\mathbf{K}$ of Equation (11) have been computed using the Control System Toolbox of MATLAB 5.1 [1,2]. Simulations of tractor rides for the different road disturbances have been performed with SIMULINK 2.0 [3].

5.1 A step disturbance at initial time

First, we choose a step disturbance (10) with $\mathbf{d_0}^T = (0, 0.1, 0, 0, 0, 0, 0, 0)$, i.e., at initial time $t_0 = 0$ the rear wheel falls off a step with a height of 0.1 m.

Figures 3 and 4 show the corresponding histories of the components of the optimization criterion of Equation (6) for the disturbance transformed into the initial value $\mathbf{x}(0) = -\mathbf{d_0}$. Less than three seconds are needed to reach an almost steady state. As expected, almost no differences are visible between the approximation provided by DIRCOL for 47 grid points (grey curves) and the optimal steady-state LQR solution (black curves) within the accuracy of the drawings. Therefore the direct transcription method provides a good approximation of the optimal trajectory using $[t_0, t_f] = [0, 5]$ for the computation and setting optimality tolerance and nonlinear feasibility tolerance [10] to 10^{-5}. The value of the performance index computed by DIRCOL is only 2.5% worse than the minimal value:

	LQR	DIRCOL
J_{active}	3.044	3.119

Also the computed estimates of the adjoint variables are relatively accurate, cf. Figure 5.

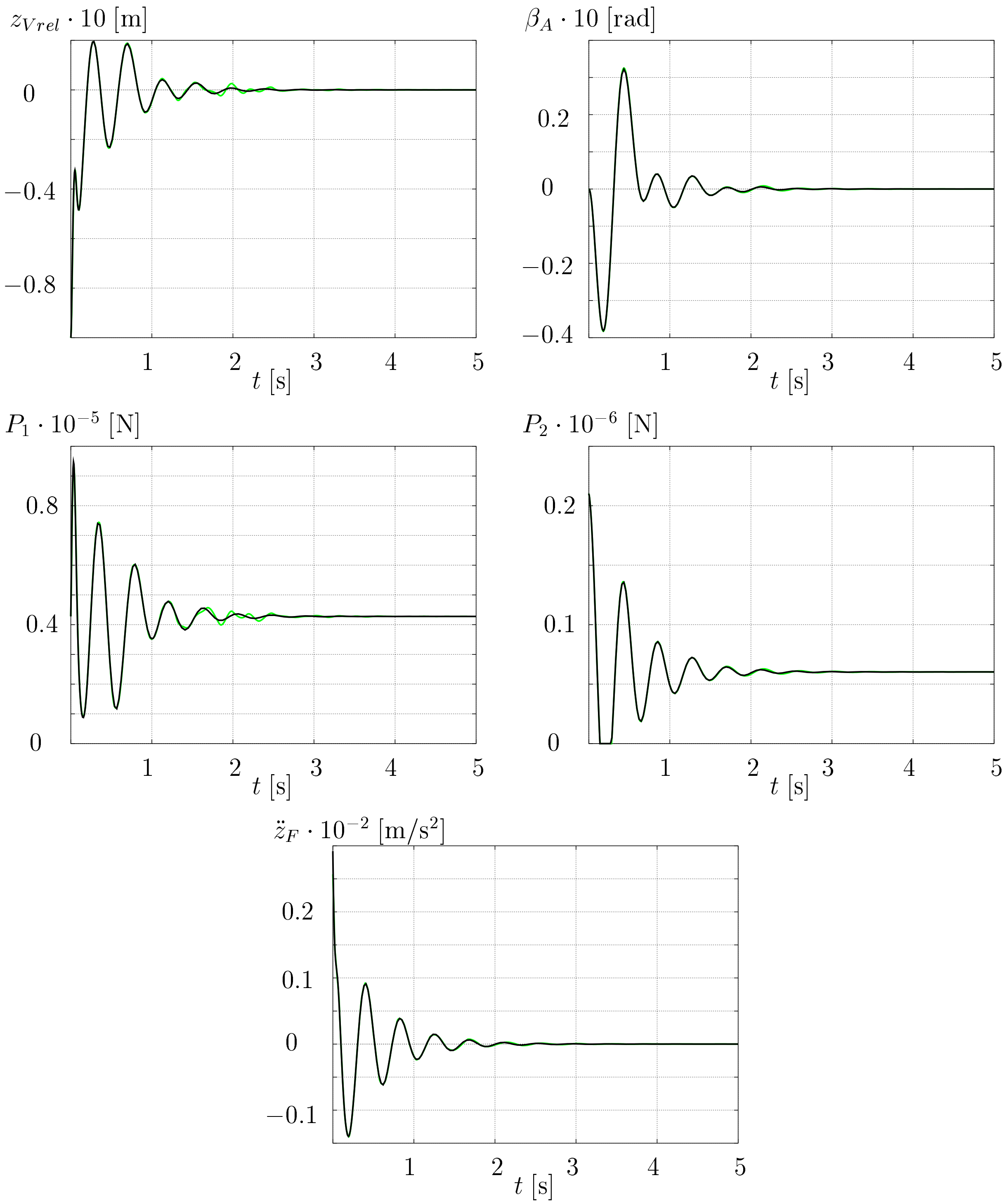

Figure 3. Histories of the state-dependent components of the objective J_{active} for the step disturbance: Approximated optimal (open-loop) solution provided by DIRCOL (grey curves) compared to the optimal (closed-loop) steady-state LQR solution (black curves).

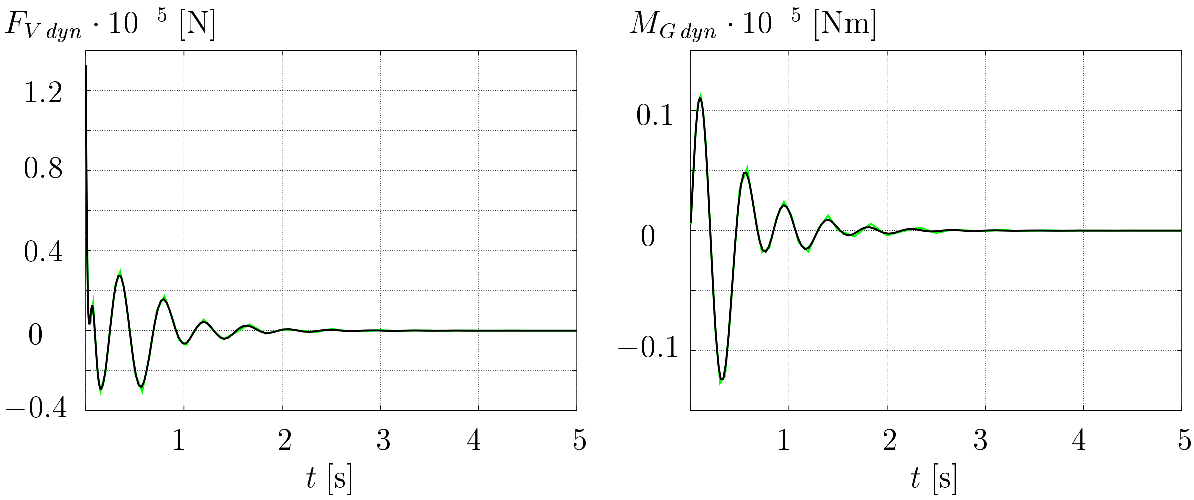

Figure 4. Histories of the approximated optimal (open-loop) controls provided by DIRCOL (grey curves, piecewise linear) compared to the optimal state-feedback controls (black curves) for the step disturbance.

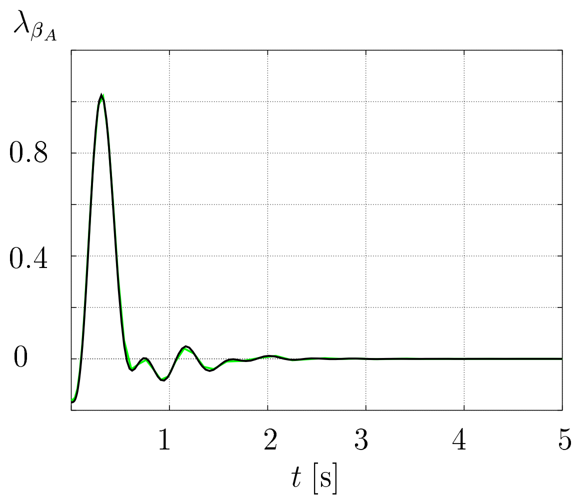

Figure 5. Estimate of the adjoint variable λ_{β_A} for the rotational motion of the implement computed by DIRCOL (grey curve) compared to the optimal steady-state LQR solution (black curve) for the step disturbance.

5.2 A ramp

The second example simulates a ride with velocity $v = 10$ km/h $= 25/9$ m/s over a ramp with height $h = 0.1$ m and length $l = 1$ m on a plane road, cf. Figure 6 and Reference 16. $\mathbf{x}(0) = \mathbf{0}$ is used as the initial value.

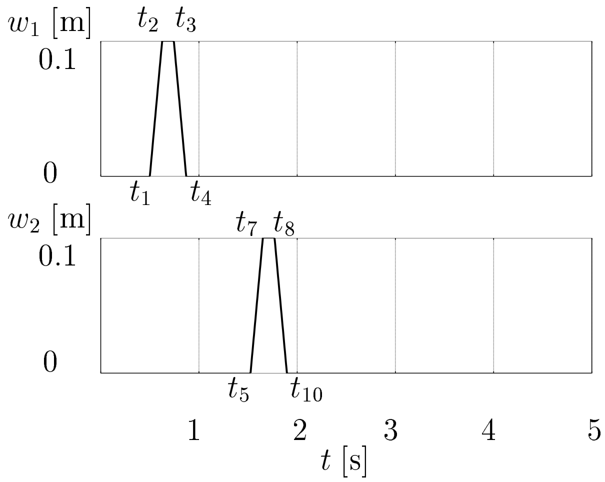

Figure 6. Disturbance signals $w_1(t)$ and $w_2(t)$ at the front wheel and the rear wheel, respectively, simulating a ride over a ramp with a height of 0.1 m and a length of 1 m at a velocity of 10 km/h.

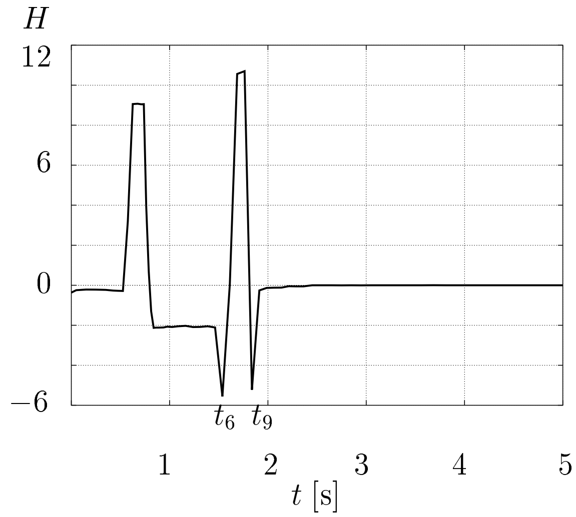

Figure 7. Estimate of the Hamiltonian computed by DIRCOL for the ramp disturbance.

The front wheel reaches the ramp at $t_1 = 0.5$ s, whereas the rear wheel reaches it with a delay of $\Delta t = 1.026$ s at t_5. For this example, the steady-state solution is not optimal because of the time-varying road disturbance $\mathbf{d}^T(t) = (w_1(t), w_2(t), 0, 0, \dot{w}_1(t), \dot{w}_2(t), 0, 0)$ with

$$w_1(t) = \begin{cases} 0\,, & 0 \leq t \leq t_1 = 0.50 \\ \frac{h}{t_2 - t_1}(t - t_1)\,, & t_1 \leq t \leq t_2 \approx 0.625 \\ h\,, & t_2 \leq t \leq t_3 \approx 0.745 \\ \frac{h}{t_2 - t_1}(t_4 - t)\,, & t_3 \leq t \leq t_4 \approx 0.871 \\ 0\,, & t_4 \leq t \leq t_f = 5.0 \end{cases}$$

and $w_2(t - \Delta t) = w_1(t)$. This can be seen also by the estimated history of the Hamiltonian computed by DIRCOL in Figure 7. For an autonomous problem, the Hamiltonian $H := \lambda^T \mathbf{f} + L$ is a constant function of time t. The piecewise definition of $w_1(t)$ and $w_2(t)$ results in a piecewise behavior of the Hamiltonian having the same switching points $t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_{10}$ (Figure 7). The two other points, t_6 and t_9, where H is not differentiable are caused by the wheel loads P_1 and P_2. When the rear wheel is driving over the ramp, first the front wheel loses contact with the road at t_6 for less than 0.1 s. Soon afterwards, at t_9 the same happens to the rear wheel. If the front wheel (or the rear wheel) leaves the road, then P_1 (or P_2) becomes zero. This causes nonlinear behavior and influences the history of the Hamiltonian.

Figures 8 and 9 show the histories of several components of the optimization criterion (6). About four seconds are needed to reach an almost steady state. The three curves in each of the pictures have the following meanings: the black dotted curve shows the solution for the passive suspension design of Eq. (4) where the parameters c_V, c_G, d_V and d_G have been optimized with respect to J_{active} of Eq. (6) (see Table 1 for the corresponding values [16]), while the grey curve is the open-loop solution provided by DIRCOL using 64 grid points and the black curve is the closed-loop LQR solution.

The main difference between the open-loop solution compared to the two closed-loop solutions for the passive suspension design and for the steady-state LQR problem is that the optimal open-loop solution uses information about the disturbance of the whole time interval. Therefore, knowing of the arrival at the ramp, the approximated optimal open-loop control acts *before* the front wheel reaches the ramp. The closed-loop solutions do not react on the ramp until it is actually reached. Consequently, the open-loop solution

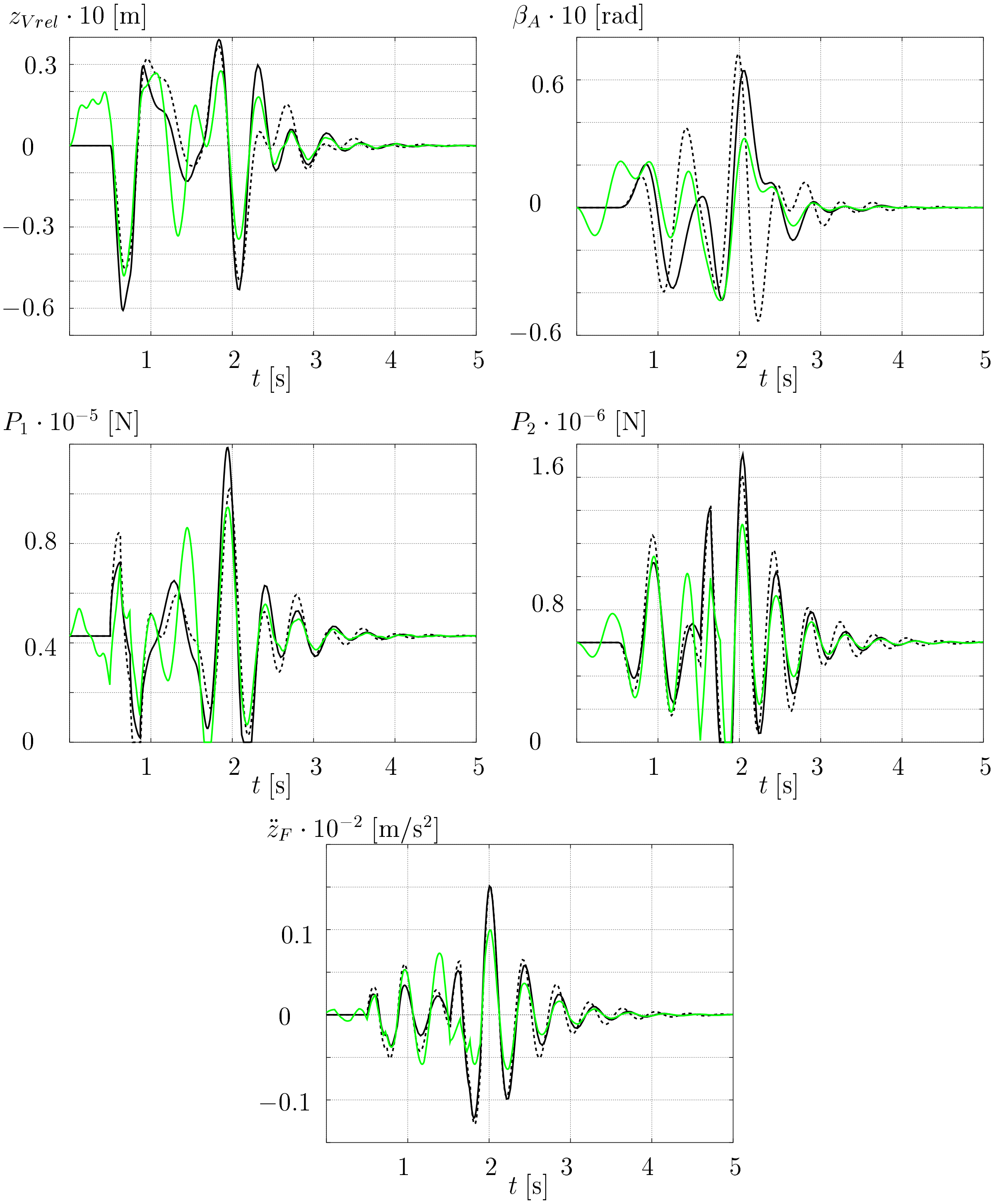

Figure 8. Histories of the state-dependent components of the objective J_{active} for the ramp disturbance: Solution for the passive suspension design (black dotted curves) compared to the no longer optimal (closed-loop) steady-state LQR solution (black curves) and the appproximated optimal (open-loop) solution provided by DIRCOL (grey curves).

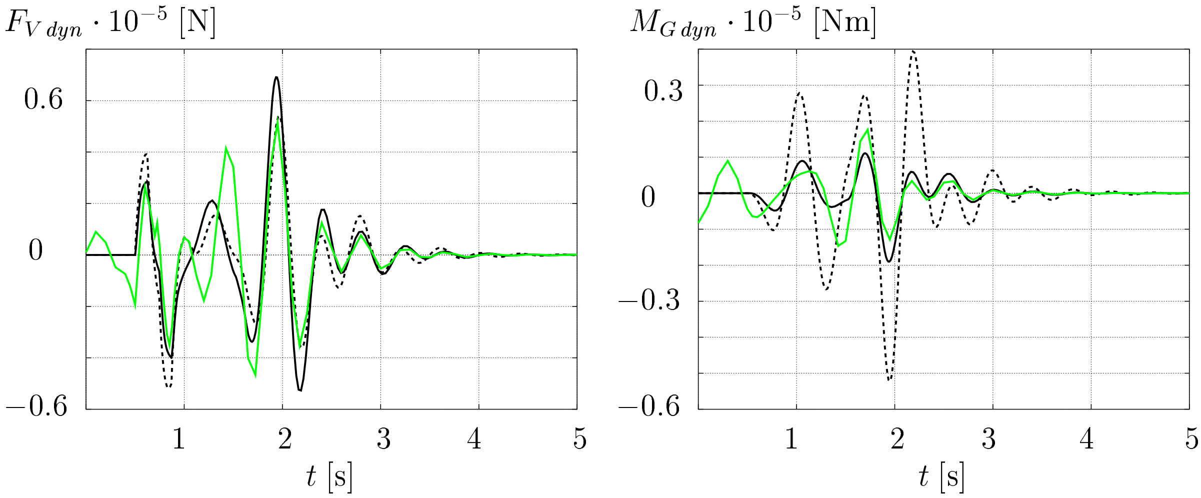

Figure 9. Histories of $F_{V\,dyn}$ and $M_{G\,dyn}$ given by Equation (4) for the passive suspension design (black dotted curves) compared to the histories of the state-feedback controls (black curves) and the approximated optimal (open-loop) controls provided by DIRCOL (grey curves, piecewise linear) for the ramp disturbance.

does not exhibit as large oscillations compared to the other two. Due to the smaller amplitudes of oscillations depicted in Figures 8 and 9, the value of the optimization criterion is essentially smaller for the approximated optimal open-loop control than for the passive suspension and the steady-state feedback control:

	passive suspension	active suspension	
		LQR	DIRCOL
J_{active}	13.060	10.251	7.751

However, the superiority of an active suspension system to the passive suspension system is obvious in both cases. In terms of the objective of Equation (6), even the in this case not optimal but state-dependent steady-state LQR solution still exhibits an overall better performance than the optimized passive suspension.

Remark 1. For selecting a suitable optimization criterion for active suspension design, simulations have been performed for different performance indices.[15,16] As briefly mentioned in Section 2, for the criterion

$$\tilde{J}_{active} = \int_{t_0}^{t_f} \Big(\mathbf{x}^T(t)\,\tilde{\mathbf{Q}}\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathbf{R}\,\mathbf{u}(t) + 2\,\mathbf{x}^T(t)\,\mathbf{S}\,\mathbf{u}(t)\Big)\ \mathrm{d}t$$

the closed-loop system is unstable. Since due to Section 4.1 the system is controllable, a solution of the steady-state LQR problem exists. But $(\mathbf{A}, \tilde{\mathbf{D}})$, where $\tilde{\mathbf{D}}$ is given by $\tilde{\mathbf{D}}^T\tilde{\mathbf{D}} = \tilde{\mathbf{Q}}$, is neither observable nor detectable which is the weakest condition for stability. This has been verified numerically using the Control System Toolbox of MATLAB. Then, the solution $\mathbf{P}$ of the

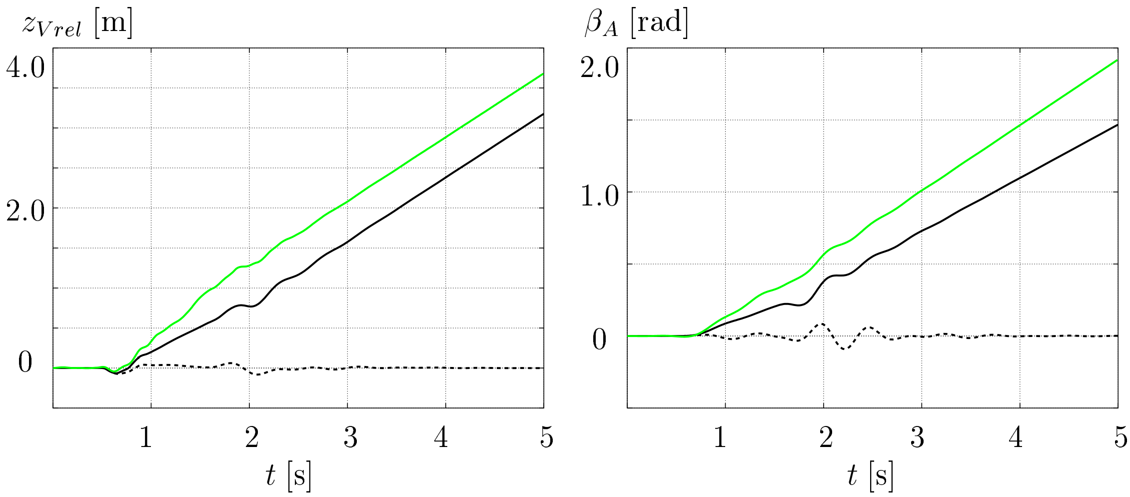

Figure 10. Histories of z_{Vrel} and β_A for the objective $\tilde{J}_{active}$ and the ramp disturbance: Stable solution for the passive suspension design (black dotted curves) compared to the steady-state LQR solution (black curves) and the approximated optimal (open-loop) solution provided by DIRCOL (grey curves) which both do not reach a steady state.

algebraic matrix Riccati equation is not symmetric, and the state β_A and the state-dependent function z_{Vrel} do not reach their steady-state value zero (Figure 10 in the case of the disturbance caused by the ramp). Applying DIRCOL to the disturbance-rejection problem with optimization criterion

$\tilde{J}_{active}$ and using 47 grid points, the computed β_A and z_{Vrel} do not reach a steady state as well (Figure 10). Both do not explicitly appear in the objective $\tilde{J}_{active}$. Therefore, z_{Vrel} and β_A have been included in the refined performance index J_{active} of Equation (6). The passive suspension design is based on the parameters c_V, c_G, d_V and d_G which have been optimized with respect to $J_{passive}$ of Equation (5).[†] It reaches the steady-state values, cf. Figure 10.

Remark 2. Generally, H^∞ methods enable a robust solution of a greater class of control problems than the LQR approach. First experiments with a minimax approach[7] didn't turn out successfully yet for the tractor problem. Besides allowing more general disturbances, the solution of nonlinear systems, where linearization techniques cannot be applied successfully, is a topic of active research in control theory.

6 Conclusions and outlook

The problem of active suspension for a planar tractor model and an objective suited for optimal active suspension have been presented. The solutions by the widely used steady-state LQR approach and a direct transcription method based on collocation and nonlinear programming have been obtained and compared for different road disturbances. The steady-state LQR solution is optimal in the case of an initial disturbance only, but provides a closed-loop solution suitable for implementation in an active suspension system of a tractor. The direct collocation method DIRCOL provides an approximation of the optimal control for general inputs from road disturbances. But its solution is in open-loop form only, and therefore is not suited for an online implementation. However, both approaches for active suspension have shown better performance than an optimized passive suspension design.

For optimal active suspension of vehicles a closed-loop form of the solution of the underlying optimal control problems having nonlinear dynamic equations, (nonlinear) constraints on the state and control variables and general objectives is needed. For an implementation in a real vehicle also the problems of incomplete state information for the design of feedback controls and of the special requirements of the devices of the real-time system have to be addressed.

A real-time capable numerical method for approximation of feedback con-

[†]The corresponding values[16] are $c_V = 6.0 \cdot 10^4$ N/m, $c_G = 7.15 \cdot 10^5$ N/m, $d_V = 3.1 \cdot 10^4$ Ns/m and $d_G = 1.9 \cdot 10^4$ Ns/m.

trols has been presented in Reference 12. The method uses successive online corrections related to a reference trajectory. In principle, it can handle quite general problems, but significant efforts are necessary to obtain the required highly accurate information about the histories of state and adjoint variables and about the switching structure of constraints as well. [12,14]

Because the direct optimization method facilitates the numerical solution of very many optimal control problems, sets of different open-loop solutions may be utilized for synthesizing an approximation of a closed-loop solution by neural networks [6] or local approximations by Taylor series [5]. This approach for synthesizing nonlinear optimal feedback controls has been introduced and tested in References 4 and 5.

Because of the increasing necessity to deal with complicated and nonlinear systems, much research has been done in the field of adaptive control within the last few years. One approach is the so-called adaptive critic (AC) method [23], which is based on Dynamic Programming (DP). In principle, DP allows the computation of optimal feedback controls even for highly nonlinear systems. But, as it is well known, the computational efforts increase exponentially with the dimension of the problem. AC methods approximate DP by incremental training of two neural network approximations of the state-dependent feedback control and of the value function of the underlying optimal control problem. The latter network approximation is called the critic. Using this approach for optimal adaptive control, e.g., of an active suspension system, is work in progress by the first author.

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